TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 355, Number 2, Pages 567–591
S 0002-9947(02)03099-4
Article electronically published on October 4, 2002

FORMULAS FOR TAMELY RAMIFIED SUPERCUSPIDAL CHARACTERS OF GL_3

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Abstract. Let F denote a p-adic local field of residual characteristic $p \neq 3$. This article gives formulas, valid on the regular elliptic set, for the irreducible supercuspidal characters of $\mathrm{GL}_3(F)$ which correspond to characters of a ramified Cartan subgroup. In the case in which F does not contain cube roots of unity, i.e., the case in which ramified cubic extensions of degree 3 over F cannot be Galois, base change results concerning "simple types" due to Bushnell and Henniart (1996) are used in the proofs.

Introduction

Let F be a nonarchimedean local field of residual characteristic $p \neq 3$, let $A|F$ be a central simple algebra of reduced degree 3, and let E be a field such that $F \subsetneq E \subset A$. Then either $A = D$, a division algebra, or $A = \mathrm{M}_3(F)$, the algebra of all 3×3 matrices. Moreover, every compact mod center Cartan subgroup of $A^\times$ is of the form $E^\times$ for some such E, and every irreducible supercuspidal representation of $A^\times$ corresponds to a quasi-character of some such $E^\times$ ([16], [6]).

In this paper we calculate formulas, valid on the regular elliptic set, for the supercuspidal characters of $A^\times$ which correspond to characters of $E^\times$ for ramified E. The reader may consult [19], in which we have discussed the case $E|F$ unramified, and [20], which deals with the more complicated case for $\mathrm{GL}_2(F)$ in which E is a wildly ramified field, i.e. $p = 2$. We do not give character values on the split torus here (see [17] for results pertaining to this problem). We mention that the character values on the regular elliptic set are sufficient to uniquely determine a supercuspidal representation (or, more generally, an arbitrary discrete series representation).

Let $D_n|F$ be a division algebra of index n. The "abstract matching theorem" of Badulescu [2], Deligne-Kazhdan-Vigneras [8], and Rogawski [18] implies the existence of a bijection between the set of irreducible representations of $D_n^\times$ and the set of essentially square-integrable representations of $\mathrm{GL}_n(F)$ which preserves the characters on the corresponding regular elliptic sets up to the sign $(-1)^{n-1}$ (cf. Theorem 1.5 below). In the tame case (i.e. $(p, n) = 1$) Moy [16] proved the existence of a bijection between these sets of representations which respects the concrete construction of the representations by Howe [12]. In general, the relationship between these two bijections is unknown, but in the case n is a prime $\neq p$ it is known that the two bijections coincide ([10]). Therefore, it suffices to determine the character

Received by the editors September 28, 1998 and, in revised form, May 2, 2002.
2000 *Mathematics Subject Classification.* Primary 22E50, Secondary 11F70.
Key words and phrases. Characters, supercuspidal, base change.

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formula on either $D_3^\times$ or $\mathrm{GL}_3(F)$ and to use the Howe-Moy bijection to obtain the formula for the corresponding representation on the other group (cf. Theorem 1.5).

Let l be an odd prime and let $L|F$ be an unramified extension of degree l. In [19] we gave the character formulas of $\mathrm{GL}_l(F)$ and $D_l^\times$ which correspond to a quasi-character of $L^\times$ (see [13] for the case $l = 2$). To obtain these results we used the fact that $L|F$ is Galois. In the present instance we have to treat the case in which $E|F$ is not Galois, and we use the base change results of Bushnell and Henniart, which concern Bushnell-Kutzko "simple characters", to transfer our results from the case $E|F$ Galois to the non-Galois case. Since the Bushnell-Henniart results apply only to the split case, we work with $\mathrm{GL}_3(F)$ and use Howe-Moy theory to deduce the corresponding $D_3^\times$ results; it is also easier to determine our character formulas near the conductor on $\mathrm{GL}_3(F)$ than on $D_3^\times$ (compare our Lemma 2.10 to the proof of Theorem 1.10 in [13]). Our main results are Theorems 2.13 and 3.9.

Let us summarize the contents of this paper, indicating its organization:

Section 1 reviews the construction of irreducible supercuspidal representations π_θ of $\mathrm{GL}_3(F)$ (of representations π'_θ of $D^\times$) from generic (see Definition 1.1) quasi-characters θ of $E^\times$. Note that π_θ is not always monomial; in the contrary case we represent its character as a $\mathbb{Q}$-linear combination of monomial characters.

Section 2 deals with the computation of the character of π_θ when $E|F$ is Galois. We represent π_θ as $\mathrm{ind}_B^{\mathrm{GL}_3(F)}\kappa_\theta$, where B is the normalizer of an Iwahori subgroup of $\mathrm{GL}_3(F)$ and κ_θ is an irreducible representation of B ("ind" denotes compact induction). We decompose the restriction $\kappa_\theta|_{E^\times}$ and use Mackey theory to calculate the characters of the constituents. Our reasoning here is reminiscent of 1.2 in [19]; we obtain the character formula except near the conductor. The rest of the formula comes directly from the explicit representation of $\kappa_\theta|_{E^\times}$ (see Lemma 2.10). This is the only new idea in the section. Results of Kutzko [15] imply that the character of κ_θ equals the character of π_θ on the elliptic set.

Section 3 treats the case in which $E|F$ is not Galois. In this case, we consider an unramified quadratic extension $L|F$; L contains cube roots of unity, $EL|L$ is Galois, and both the methods and results of Section 2 apply to $\mathrm{GL}_3(L)$. By Bushnell and Henniart [4] we have a base change lift κ_L of $\kappa_\theta|_{B^1}$ to B_L^1 such that the twisted trace of κ_L by a generator τ of $\mathrm{Gal}(L|F)$ gives the trace of κ_θ (see Proposition 3.3, and see above Proposition 3.3 for the notations B^1 and B_L^1). Note that we do not have to assume that the characteristic of F is zero, since we do not use the Arthur-Clozel base change lift ([1]). We calculate the twisted trace of κ_L much as in Section 2; we state our main result as Theorem 3.9. This is the interesting application of the Bushnell-Henniart base change lift.

Our final character formulas seem simple and beautiful. As in the unramified case, the analog of Weyl's character formula holds. By contrast we know that the formulas for wildly ramified characters are more complicated (see [20]).

To conclude this introduction, we compare our formula with the result of [7]. The same type of character formula for the division algebra of reduced degree l was given by Corwin, Moy and Sally, Jr. [7]. Their character formula contains some Gauss sum associated with a quadratic form. They have only shown that this Gauss sum is a fourth root of unity when $p \neq 2$. In this paper, we treat only the case $l = 3$, but we have determined the root number completely, including the case $p = 2$ in sections 2 and 3. Moreover we find that the Kloosterman sum appears in the character formula (cf. Lemma 2.10). This simplifies Theorem 4.1 (d) in [7].

Notation. Let F be a nonarchimedean local field. We denote by $\mathcal{O}_F$, P_F, ϖ_F, k_F and v_F the maximal order of F, the maximal ideal of $\mathcal{O}_F$, a prime element of P_F, the residue field of F and the valuation of F normalized by $v_F(\varpi_F) = 1$. Let $q = |k_F|$. When A is a division algebra over F, the reduced trace of A to F is denoted Trd. The usual matrix trace is denoted by Tr. Hereafter we fix an additive character ψ of F whose conductor is P_F, i.e., ψ is trivial on P_F and not trivial on $\mathcal{O}_F$. For an extension E over F, we denote by tr_E, n_E the trace and norm to F respectively. We set $\psi_E = \psi \circ \mathrm{tr}_E$. For an irreducible admissible representation π of $A^\times$, the conductoral exponent of π is defined to be the integer $f(\pi)$ such that the local constant $\varepsilon(s, \pi, \psi)$ of Godement and Jacquet [9] is of the form $aq^{-s(f(\pi)-3)}$. We call π *minimal* if

$$f(\pi) = f_{\min}(\pi) = \min_{\eta} f(\pi \otimes (\eta \circ \mathrm{Nr})),$$

where η runs through the set of quasi-characters of $F^\times$. Let G be a totally disconnected, locally compact group. We denote by $\widehat{G}$ the set of (equivalence classes of) irreducible admissible representations of G. For a closed subgroup H of G and a representation ρ of H, we denote by $\mathrm{Ind}_H^G \rho$ (resp. $\mathrm{ind}_H^G \rho$) the induced representation (resp. compactly induced representation) of ρ to G. For a representation π of G, we denote by $\pi|_H$ the restriction of π to H.

1. Construction of the representation

Let E be a ramified extension of F of degree 3. E can be embedded in A, and the embedding is unique up to conjugacy. In this section, we recall the construction of the supercuspidal representation of $A^\times$ from the quasi-character of $E^\times$.

Definition 1.1. Let θ be a quasi-character of $E^\times$ and $f(\theta) = \min\{n | \operatorname{Ker}\theta \supset 1 + P_E^n\}$. θ is called generic if $f(\theta) \not\equiv 1 \mod 3$.

We write $\widehat{E^\times}_{\mathrm{gen}}$ for the set of generic quasi-characters of $E^\times$. If $\theta \in \widehat{E^\times}_{\mathrm{gen}}$, then there exists $\gamma_\theta \in P_E^{1-f(\theta)} - P_E^{2-f(\theta)}$ such that $F(\gamma_\theta) = E$ and

$$\theta(1+x) = \psi(\mathrm{tr}_{E|F}(\gamma_\theta x)) \quad \text{for all} \quad x \in P_E^{[(f(\theta)+1)/2]}. \tag{1.1}$$

Let us begin the construction of an irreducible supercuspidal representation of $A^\times = \mathrm{GL}_3(F)$ corresponding to a generic character of $E^\times$. We choose $\theta \in \widehat{E^\times}_{\mathrm{gen}}$ such that $f(\theta) = n + 1 \not\equiv 1$. We assume that the prime element $\varpi_E \in E$ satisfies $\varpi_E^3 = \varpi_F$; using the F-basis $\{\varpi_E^2, \varpi_E, 1\}$ of E, we identify A with $\mathrm{End}_F(E)$ and $G = \mathrm{GL}_3(F)$ with $\mathrm{Aut}_F(E)$. Using the lattice flag $\{P_E^i\}_{i\in\mathbb{Z}}$, we construct the maximal compact modulo center subgroup which contains $E^\times$:

Definition 1.2. For $i \in \mathbb{Z}$, set $A^i = \{f \in \mathrm{M}_3(F) \mid f(P_E^j) \subset P_E^{j+i} \text{ for all } j \in \mathbb{Z}\}$. Put $K = (A^0)^\times, B = E^\times K$ and $K^i = 1 + A^i$ for $i \geq 1$.

We note that K is an Iwahori subgroup of $\mathrm{GL}_3(F)$ and B is its normalizer. At the first step toward the construction of a supercuspidal character of G from θ, we construct a representation of B. We have $\gamma = \gamma_\theta \in P_E^{-n}$ such that $\theta(1+x) = \psi_E(\gamma x)$ for all $x \in P_E^m$ $(m = [(n+2)/2])$; we define a character ψ_γ of K^m by setting $\psi_\gamma(1+x) = \psi(\mathrm{Trd}(\gamma x))$ for $x \in A^m$. Set $H = E^\times K^m$ and define a quasi-character ρ_θ of H by setting

$$\rho_\theta(h \cdot g) = \theta(h)\psi_\gamma(g) \qquad \text{for} \quad h \in E^\times, \quad g \in K^m. \tag{1.2}$$

Let J denote the stabilizer of ψ_γ in B, i.e.,

$$J = \{a \in B \mid \psi_\gamma(axa^{-1}) = \psi_\gamma(x) \quad \text{for} \quad x \in K^m\}.$$

Then $J = E^\times K^{m'}$, where $m' = [(n+1)/2]$.

If $n+1 = 2m$ is even, then $J = H = E^\times K^m$. By Clifford theory, $\mathrm{Ind}_H^B \rho_\theta$ is, in this case, an irreducible representation of B. We set

$$\kappa_\theta = \mathrm{Ind}_H^B \rho_\theta. \tag{1.3}$$

If $n+1 = 2m-1$ is odd, then $J = E^\times K^{m-1}$; so we have to determine an irreducible component of $\mathrm{Ind}_H^J \rho_\theta$. For a subgroup $M \subset B$, we write $M^1 = M \cup F^\times K$. In particular, $H^1 = F^\times(1+P_E)K^m$. It is known and not difficult to show that

$$(\mathrm{Ind}_H^J \rho_\theta)|_{J^1} = \mathrm{Ind}_{H^1}^{J^1}(\rho_\theta|_{H_1}) = q\eta. \tag{1.4}$$

This η can be extended to J in $|E^\times/F^\times(1+P_E)| = 3$ ways. To determine the extension by θ, we will express it by a linear combination of $\mathrm{Ind}_H^J \rho_{\theta\otimes\chi}$ ($\chi \in (E^\times/F^\times(1+P_E))\hat{}$).

Lemma 1.3. *Define the virtual representation κ_θ of J by*

$$\kappa_\theta = \begin{cases} \dfrac{1-q}{3q} \displaystyle\sum_{\chi \in (E^\times/F^\times(1+P_E))\hat{}} \mathrm{Ind}_H^B \rho_{\theta\otimes\chi} + \mathrm{Ind}_H^B \rho_\theta & \text{if } q \equiv 1 \mod 3, \\ \dfrac{1+q}{3q} \displaystyle\sum_{\chi \in (E^\times/F^\times(1+P_E))\hat{}} \mathrm{Ind}_H^B \rho_{\theta\otimes\chi} - \mathrm{Ind}_H^B \rho_\theta & \text{if } q \equiv 2 \mod 3. \end{cases} \tag{1.5}$$

Then κ_θ is a real representation and an irreducible component of $\mathrm{Ind}_H^B \rho_\theta$.

Proof. Let $\{\eta_1, \eta_2, \eta_3\}$ be the set of the extensions of η to J and $(E^\times/F^\times(1+P_E))\hat{} = \{\chi_1, \chi_2, \chi_3\}$. It follows from Lemma 3.5.35 in [16] that

$$\mathrm{Ind}_H^J \rho_\theta = \frac{\left(q - \left(\frac{q}{3}\right)\right)}{3} \sum_{i=1}^{3} \eta_i + \left(\frac{q}{3}\right) \eta_j \tag{1.6}$$

for a unique j, where $\left(\frac{q}{3}\right)$ is the Legendre symbol. Let us denote η_θ by this η_j. From this irreducible decomposition of $\mathrm{Ind}_H^J \rho_\theta$, we have

$$\begin{pmatrix} \mathrm{Ind}_H^J \rho_{\theta\otimes\chi_1} \\ \mathrm{Ind}_H^J \rho_{\theta\otimes\chi_2} \\ \mathrm{Ind}_H^J \rho_{\theta\otimes\chi_3} \end{pmatrix} = \left(\left(\frac{q}{3}\right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\left(q - \left(\frac{q}{3}\right)\right)}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right) \begin{pmatrix} \eta_{\theta\otimes\chi_1} \\ \eta_{\theta\otimes\chi_2} \\ \eta_{\theta\otimes\chi_3} \end{pmatrix}.$$

By the Clifford theory, $\kappa_\theta = \mathrm{Ind}_J^B \rho_\theta$ is an irreducible representation of B. Thus we obtain the desired formula for κ_θ by calculating the inverse of the coefficient matrix. □

The following result is well-known. (See [16].)

Theorem 1.4. *Let the notation be as above. Then κ_θ is an irreducible representation of B. Put $\pi_\theta = \mathrm{ind}_B^G \kappa_\theta$. Then π_θ is an irreducible supercuspidal representation of G such that:*

1. *the L-function of π_θ is 1;*
2. *$\varepsilon(\pi_\theta, \psi) = \varepsilon(\theta, \psi_E)$; in particular $f(\pi_\theta) = f(\theta) + 3$.*

Any irreducible supercuspidal representation π of G such that $f(\pi) \not\equiv 0 \bmod 3$ can be written in the form $\pi = \pi_\theta$ for some $\theta \in \widehat{E}^\times_{\mathrm{gen}}$, where E is a ramified extension of degree 3 over F.

Next we construct an irreducible representation of $D^\times$ from $\theta \in E^\times_{\mathrm{gen}}$. Let $f(\theta) = n+1$. We recall that $n \not\equiv 0 \bmod 3$. We define a function ψ_γ on $1 + P_D^m$ by $\psi_\gamma(1+x) = \psi(\mathrm{Trd}(\gamma x))$ for $x \in P_D^m$. Then ψ_γ is a quasi-character of $1 + P_D^m$. Set $H' = E^\times(1+P_D^m) \subset D^\times$ and define a quasi-character ρ'_θ of H' by

$$\rho'_\theta(h \cdot g) = \theta(h)\psi_\gamma(g) \qquad \text{for} \quad h \in E^\times, \quad g \in 1 + P_D^m. \tag{1.7}$$

When $n+1$ is even, i.e. $n+1 = 2m$, we set

$$\pi'_\theta = \mathrm{Ind}_{H'}^{D^\times} \rho'_\theta. \tag{1.8}$$

When $n+1$ is odd, i.e. $n+1 = 2m-1$, $\mathrm{Ind}_{H'}^{D^\times} \rho'_\theta$ is not irreducible. As in the GL_3 case, we can take its irreducible component as a $\mathbb{Q}$-linear combination of $\mathrm{Ind}_{H'}^{D^\times} \rho'_{\theta\otimes\chi}$ $(\chi \in (E^\times/F^\times(1+P_E))\hat{\ })$.

Lemma 1.5. *Define the virtual representation π'_θ of $D^\times$ by*

$$\pi'_\theta = \begin{cases} \dfrac{1-q}{3q} \displaystyle\sum_{\chi \in (E^\times/F^\times(1+P_E))\hat{}} \mathrm{Ind}_{H'}^{D^\times} \rho'_{\theta\otimes\chi} + \mathrm{Ind}_{H'}^{D^\times} \rho'_\theta & \text{if} \quad q \equiv 1 \mod 3, \\ \dfrac{1+q}{3q} \displaystyle\sum_{\chi \in (E^\times/F^\times(1+P_E))\hat{}} \mathrm{Ind}_{H'}^{D^\times} \rho'_{\theta\otimes\chi} - \mathrm{Ind}_{H'}^{D^\times} \rho'_\theta & \text{if} \quad q \equiv 2 \mod 3. \end{cases} \tag{1.9}$$

Then π'_θ is a real representation and an irreducible component of $\mathrm{Ind}_{H'}^{D^\times} \rho'_\theta$.

Proof. The argument from Lemma 3.5.28 to Lemma 3.5.35 in [16] can be applied to the division algebra case. Therefore we can prove this lemma in the same way as Lemma 1.3. □

The following result is essentially well-known. (See [3] and [16].)

Theorem 1.6. *Let the notation be as above. Then π'_θ is an irreducible minimal representation of $D^\times$ such that:*

1. *the L-function of π'_θ is 1;*
2. *$\varepsilon(\pi'_\theta, \psi) = \varepsilon(\theta, \psi_E)$; in particular $f(\pi'_\theta) = f(\theta) + 3$.*

Any irreducible representation π' of $D^\times$ such that $f(\pi') \not\equiv 0 \mod 3$ can be written in the form $\pi' = \pi'_\theta$ for some $\theta \in \widehat{E}^\times_{\mathrm{gen}}$, where E is a ramified extension of degree 3 over F.

Now we define the correspondence $\pi'_\theta \leftrightarrow \pi_\theta$ by parameterizing in each case the set of representations by the set of generic quasi-characters of $E^\times$ (Howe's bijection [16]).

Proposition 1.7. *The characters of π_θ and π'_θ are equal on regular elliptic conjugacy classes, which are identified between the groups $D^\times$ and $\mathrm{GL}_3(F)$.*

Remark. The set of irreducible supercuspidal representations of $A^\times$ with minimal conductor $\equiv 0 \mod 3$ is parameterized by the set of regular quasi-characters of $L^\times$, where L is an unramified extension of F of degree 3. The character formula for such a representation on the regular elliptic set was given in [19].

In concluding this section we reformulate a result of Kutzko ([15]) to put it into the precise form in which we shall apply it:

Theorem 1.8. *Let x be a regular elliptic element of G.*

1. If $F(x)|F$ is ramified and $x \notin F^\times(1+P_{F(x)}^{n+1})$, then

$$\chi_{\pi_\theta}(x) = \chi_{\kappa_\theta}(x).$$

2. If $F(x)|F$ is unramified and $x \notin F^\times(1+P_{F(x)}^{[(n+3)/3]})$, then

$$\chi_{\pi_\theta}(x) = 0.$$

Proof. It is obtained by applying Proposition 5.5 in [15] to our case. □

2. Character formula for the Galois case

In this section, we treat the case in which F contains a primitive cube root ζ of unity. Since $p \neq 3$, we have $q = p^f \equiv 1 \mod 3$; in this case all ramified extensions of degree 3 of F are Galois. We shall calculate a formula only for the character of κ_θ; by applying Theorem 1.8 we obtain from this calculation a formula for the character of π_θ. As in the first section, we identify $\mathrm{M}_3(F)$ with $\mathrm{End}_F(E)$ by the F-basis $\{\varpi_E^2, \varpi_E, 1\}$ of E. Thus we get the explicit matrix forms of various objects:

$$\varpi_E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varpi_F & 0 & 0 \end{pmatrix}, \tag{2.1}$$

$$E = \left\{ \begin{pmatrix} a & b & c \\ c\varpi_F & a & b \\ b\varpi_F & c\varpi_F & a \end{pmatrix} \middle| \, a, b, c \in F \right\}, \tag{2.2}$$

$$K = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \middle| \begin{array}{ll} a_{ij} \in \mathcal{O}_F & \text{if } i < j \\ a_{ii} \in \mathcal{O}_F^\times & \\ a_{ij} \in P_F & \text{if } i > j \end{array} \right\}, \tag{2.3}$$

$$A^0 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \middle| \begin{array}{ll} a_{ij} \in \mathcal{O}_F & \text{if } i \leq j \\ a_{ij} \in P_F & \text{if } i > j \end{array} \right\}, \tag{2.4}$$

$$A^1 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \middle| \begin{array}{ll} a_{ij} \in \mathcal{O}_F & \text{if } i < j \\ a_{ij} \in P_F & \text{if } i \geq j \end{array} \right\}. \tag{2.5}$$

Let σ be the generator of $\mathrm{Gal}(E|F)$ such that ${}^\sigma\varpi_E = \varpi_E\zeta$.

Lemma 2.1. *Put*

$$\xi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta \end{pmatrix}.$$

Then ξ satisfies $\xi^3 = 1$,

$$\xi x \xi^{-1} = {}^\sigma x \quad \textit{for any} \quad x \in E,$$

and

$$\begin{array}{rcccccl} \mathrm{M}_3(F) & = & E & \oplus & E\xi & \oplus & E\xi^2, \\ A^0 & = & \mathcal{O}_E & \oplus & \mathcal{O}_E\xi & \oplus & \mathcal{O}_E\xi^2, \\ A^1 & = & P_E & \oplus & P_E\xi & \oplus & P_E\xi^2, \\ A^2 & = & P_E^2 & \oplus & P_E^2\xi & \oplus & P_E^2\xi^2. \end{array} \tag{2.6}$$

Proof. This is obvious from the above explicit matrix forms of ϖ_E and A^i. □

Let θ be a generic quasi-character of $E^\times$ with $f(\theta) = n+1$. If n is odd, then $n+1 = 2m$ and $\kappa_\theta = \mathrm{Ind}_H^B \rho_\theta$, where $H = E^\times(1+A^m)$. By Mackey theory ([22], Proposition 22),

$$\kappa_\theta|_{E^\times} = \bigoplus_{a \in H\backslash B/E^\times} \mathrm{Ind}_{a^{-1}Ha\cap E^\times}^{E^\times} {}^a\rho_\theta, \tag{2.7}$$

where ${}^a\rho_\theta(x) = \rho_\theta(axa^{-1})$ for $x \in a^{-1}Ha \cap E^\times$. If n is even, then $n+1 = 2m-1$ and

$$\begin{aligned}\kappa_\theta|_{E^\times} = \frac{1-q}{3q} &\sum_{\chi \in (E^\times/F^\times(1+P_E))^\wedge} \sum_{a \in H\backslash B/E^\times} \mathrm{Ind}_{a^{-1}Ha\cap E^\times}^{E^\times} {}^a\rho_{\theta\otimes\chi} \\ &+ \sum_{a\in H\backslash B/E^\times} \mathrm{Ind}_{a^{-1}Ha\cap E^\times}^{E^\times} {}^a\rho_\theta,\end{aligned} \tag{2.8}$$

since $q \equiv 1 \mod 3$.

For simplicity, we fix a generic character θ of $E^\times$ with $f(\theta) = n+1$ and drop the subscript θ in the representations determined by θ, e.g. $\rho = \rho_\theta$, $\kappa = \kappa_\theta$ and so on.

First we give a complete system of representatives for the set of the double cosets $H\backslash B/E^\times$.

Lemma 2.2. 1. *Let $a = 1+\alpha_1\xi+\alpha_2\xi^2$ for $\alpha_1, \alpha_2 \in \mathcal{O}_E$. Then $a \in K$ is equivalent to $1+\alpha_1^3+\alpha_2^3-3\alpha_1\alpha_2 \notin P_E$.*

2. *Let $a = 1+\alpha_1\xi+\alpha_2\xi^2, b = 1+\beta_1\xi+\beta_2\xi^2 \in K$. Then $Ha = Hb$ if and only if $\alpha_i - \beta_i \in P_E^m$ for $i = 1, 2$.*

Proof. 1. Since a is obviously in A^0, it is only necessary to determine necessary and sufficient conditions such that the diagonal entries belong to $\mathcal{O}_F^\times$, i.e., necessary and sufficient conditions that the product of the diagonal entries belong to $\mathcal{O}_F^\times$. Using explicit representations for the diagonal entries, multiplying them and simplifying by using the relation $0 = 1+\zeta+\zeta^2$, the reader may verify the given condition.

2. By 2.6, $Ha = Hb$ implies that there exist $\gamma_0 \in \mathcal{O}_E^\times$ and $\gamma_1, \gamma_2 \in P_E^m$ such that $b = (\sum_{i=0}^2 \gamma_i\xi^i)a$. Since $A^0 = \mathcal{O}_E \oplus \mathcal{O}_E\xi \oplus \mathcal{O}_E\xi^2$ and $\xi x = {}^\sigma x\xi$ for $x \in E$, we obtain

$$\begin{aligned}1 &= \gamma_0 + (\gamma_1{}^\sigma\alpha_2 + \gamma_2{}^{\sigma^2}\alpha_1),\\ \beta_1 - \alpha_1 &= (\gamma_0-1)\alpha_1 + \gamma_1 + \gamma_2{}^{\sigma^2}\alpha_2,\\ \beta_2 - \alpha_2 &= (\gamma_0-1)\alpha_2 + \gamma_2 + \gamma_1{}^\sigma\alpha_1.\end{aligned}$$

Therefore we have $\gamma_0 \in 1+P_E^m$ and $\beta_i - \alpha_i \in P_E^m \quad (i=1,2)$.

Conversely we assume $\beta_i - \alpha_i \in P_E^m \quad (i=1,2)$. Put $ba^{-1} = \sum_{i=0}^2 \gamma_i\xi^i$. By eliminating γ_0 from the above three equations, we obtain

$$\begin{aligned}\beta_1 - \alpha_1 &= (1-\alpha_1{}^\sigma\alpha_2)\gamma_1 + ({}^{\sigma^2}\alpha_2 - \alpha_1{}^{\sigma^2}\alpha_1)\gamma_2,\\ \beta_2 - \alpha_2 &= ({}^\sigma\alpha_1 - \alpha_2{}^\sigma\alpha_2)\gamma_1 + (1 - {}^{\sigma^2}\alpha_1\alpha_2)\gamma_2.\end{aligned}$$

Noting that $\alpha_i - \beta_i \in P_E^m$, we use Cramer's Rule to verify that $\gamma_i \in P_E^m$ $(i=1,2)$, i.e., we check that the determinant of the coefficient matrix for the system of two equations in the variables γ_1 and γ_2 belongs to $\mathcal{O}_E^\times$. A routine calculation reveals that this determinant equals $\det(a)$, as given in Lemma 2.5 below; clearly, $\det(a) \in \mathcal{O}_E^\times$. Since $\det(a)$ is the denominator for Cramer's Rule and $\alpha_i - \beta_i \in P_E^m$ $(i=1,2)$

imply that the numerator belongs to P_E^m, we see that $\gamma_1, \gamma_2 \in P_E^m$. From the three preceding equations we then see that $\gamma_0 \in 1 + P_E^m$ too. □

In order to give a set of representatives for $H\backslash B/E^\times$ we introduce some more notation. Put $\mathcal{O}_E^{(1)} = \operatorname{Ker} n_E$ and

$$M = \{({}^\sigma\alpha\alpha^{-1}, {}^{\sigma^2}\alpha\alpha^{-1}) \mid \alpha \in E^\times\} \subset \mathcal{O}_E^{(1)} \times \mathcal{O}_E^{(1)}.$$

For $0 < \mu < m$, we set

$$\begin{aligned} I_{\mu,1} &= (1 + P_E^{m-\mu}) \times P_E^m \backslash \varpi_E^\mu \mathcal{O}_E^\times \times P_E^\mu / M, \\ I_{\mu,2} &= P_E^m \times (1 + P_E^{m-\mu}) \backslash P_E^{\mu+1} \times \varpi_E^\mu \mathcal{O}_E^\times / M, \\ J_{\mu,1} &= \left(\varpi_F^\mu n_E(\mathcal{O}_E^\times)/(1 + P_F^{[(m-\mu+2)/3]})\right) \times \left(P_E^{2\mu}/P_E^{m+\mu}\right), \\ J_{\mu,2} &= \left(P_E^{2\mu+1}/P_E^{m+\mu}\right) \times \left(\varpi_F^\mu n_E(\mathcal{O}_E^\times)/(1 + P_F^{[(m-\mu+2)/3]})\right). \end{aligned}$$

When $\mu = 0$, we set

$$\begin{aligned} I_{0,1} &= (1 + P_E^m) \times P_E^m \backslash \{(\beta_1, \beta_2) \in \mathcal{O}_E^\times \times \mathcal{O}_E | 1 + \beta_1^3 + \beta_2^3 - 3\beta_1\beta_2 \notin P_E\}/M, \\ I_{0,2} &= P_E^m \times (1 + P_E^m) \backslash \{(\beta_1, \beta_2) \in P_E \times \mathcal{O}_E^\times | 1 + \beta_2^3 \notin P_E\}/M, \\ J_{0,1} &= \Big\{(\beta_1, \beta_2) \in (n_E(\mathcal{O}_E^\times)/1 + P_F^{[(m+2)/3]}) \times (\mathcal{O}_E/P_E^m) \ \Big| \\ &\qquad\qquad 1 + \beta_1 + \beta_1^{-1}\beta_2^3 - 3\beta_2 \notin P_E\Big\}, \\ J_{0,2} &= \Big\{(\beta_1, \beta_2) \in (P_E/P_E^m) \times (n_E(\mathcal{O}_E^\times)/(1 + P_F^{[(m+2)/3]})) \ \Big| \ 1 + \beta_2 \notin P_E\Big\}. \end{aligned}$$

For $0 \leq \mu < m$, we set

$$\tilde{I}_{\mu,i} = \{1 + \beta_1\xi + \beta_2\xi^2 | (\beta_1, \beta_2) \in I_{\mu,i}\}. \tag{2.9}$$

Lemma 2.3. 1. *A complete set of representatives of the double coset $H\backslash B/E^\times$ is given by*

$$\begin{aligned} &\{1, \xi, \xi^2\} \cup \bigcup_{\mu=0}^{m-1} (\tilde{I}_{\mu,1} \cup \tilde{I}_{\mu,2}) \cup \bigcup_{\mu=1}^{m-1} (\tilde{I}_{\mu,1}\xi \cup \tilde{I}_{\mu,2}\xi) \\ &\cup \bigcup_{\mu=1}^{m-1} \tilde{I}_{\mu,1}\xi^2 \cup \bigcup_{\mu=0}^{m-1} \tilde{I}_{\mu,2}\xi^2. \end{aligned} \tag{2.10}$$

2. *Let $\varphi_1(\beta_1, \beta_2) = (n_E(\beta_1), \beta_1{}^\sigma\beta_2)$ for $(\beta_1, \beta_2) \in \varpi_E^\mu \mathcal{O}_E^\times \times P_E^\mu$ and $\varphi_2(\beta_1, \beta_2) = (\beta_2{}^{\sigma^2}\beta_1, n_E(\beta_2))$ for $(\beta_1, \beta_2) \in P_E^{\mu+1} \times \varpi_E^\mu \mathcal{O}_E^\times$. Then φ_i induces a bijection from $I_{\mu,i}$ to $J_{\mu,i}$ for $i = 1, 2$.*

Proof. Since $H(1 + \beta_1\xi + \beta_2\xi^2)\alpha = H(1 + \beta_1{}^\sigma\alpha\alpha^{-1}\xi + \beta_2{}^{\sigma^2}\alpha\alpha^{-1}\xi^2)$ for any $\alpha \in E^\times$ and any element of $B/E^\times$ has a representative of the form $(1 + \beta_{i1}\xi + \beta_{i2}\xi^2)\xi^i$ $(0 \leq i \leq 2)$, where $\beta_{01}, \beta_{02}, \beta_{22} \in \mathcal{O}_E$ and $\beta_{11}, \beta_{12}, \beta_{21} \in P_E$, the first part of the lemma follows from Lemma 2.2. To prove the second part of the lemma, it suffices to see that φ_1 induces a bijection from $I_{0,1}$ to $J_{0,1}$. Let $\beta_1, \gamma_1 \in \mathcal{O}_E^\times$ and $\beta_2, \gamma_2 \in \mathcal{O}_E$. If (γ_1, γ_2) belongs to the double coset $((1 + P_E^m) \times P_E^m)(\beta_1, \beta_2)M$, then there exist $\alpha \in \mathcal{O}_E^\times$ and $y_1, y_2 \in P_E^m$ such that

$$\gamma_1 = {}^{\sigma}\alpha\alpha^{-1}\beta_1(1+y_1),$$
$$\gamma_2 = {}^{\sigma^2}\alpha\alpha^{-1}(\beta_2+y_2).$$

This implies

$$n_E(\beta_1) \equiv n_E(\gamma_1) \mod 1+P_E^m \quad \text{(multiplicative equivalence)},$$
$$\gamma_1{}^{\sigma}\gamma_2 \equiv \beta_1{}^{\sigma}\beta_2 \mod P_E^m.$$

Therefore φ_1 induces a well-defined map from $I_{0,1}$ to $J_{0,1}$. The induced map's bijectivity follows from the bijectivity of the map

$$\mathcal{O}_E^{(1)}\backslash\mathcal{O}_E^\times/(1+P_E^j) \xrightarrow{n_E} n_E(\mathcal{O}_E^\times)/(1+P_F^{[(j+2)/3]}).$$

□

Next we consider the term $a^{-1}Ha \cap E^\times$.

Lemma 2.4. *Let $a \in \tilde{I}_{\mu,j}$ ($j = 1$ or 2 and $0 \le \mu < m$). Then $a^{-1}Ha \cap E^\times = F^\times(1+P_E^{m-\mu})$.*

Proof. Let $a = 1+\beta_1\xi+\beta_2\xi^2 \in \tilde{I}_{\mu,j}$ and let $\alpha \in a^{-1}Ha \cap E^\times$. Let us begin by showing that $\alpha \in F^\times(1+P_E^{m-\mu})$. First, since $F^\times \subset E^\times$, we may assume that $0 \le v_E(\alpha) \le 2$. Since $a\alpha = \gamma a$, where $\gamma = \gamma_0+\gamma_1\xi+\gamma_2\xi^2 \in H$, we have $\gamma_i \in E^\times$ ($i = 0,1,2$) such that $\gamma_0^{-1}\gamma_i \in P_E^m$ ($i = 1,2$) and $v_E(\gamma_0) = v_E(\alpha)$. From this we see that

$$\gamma_0 = \alpha - (\gamma_1{}^{\sigma}\beta_2 + \gamma_2{}^{\sigma^2}\beta_1),$$
$$({}^{\sigma}\alpha - \gamma_0)\beta_1 = \gamma_1 + \gamma_2{}^{\sigma^2}\beta_2, \tag{2.11}$$
$$({}^{\sigma^2}\alpha - \gamma_0)\beta_2 = \gamma_2 + \gamma_1{}^{\sigma}\beta_1.$$

Replacing γ_0 by $\alpha - (\gamma_1{}^{\sigma}\beta_2 + \gamma_2{}^{\sigma^2}\beta_1)$, we obtain the relations

$$({}^{\sigma^i}\alpha - \alpha)\beta_i \in P_E^{m+v_E(\alpha)} \qquad (i = 1,2).$$

If $v_E(\alpha) = 1$ or 2, then $v_E({}^{\sigma^i}\alpha - \alpha) = v_E(\alpha)$, which implies that $v_E(\beta_i) \ge m$. From this we have a contradiction to the hypothesis that $\mu < m$; so we may assume that $v_E(\alpha) = 0$. In this case, we see that $v_E({}^{\sigma^i}\alpha - \alpha) \ge m-\mu$, and we therefore conclude that $\alpha \in F^\times(1+P_E^{m-\mu})$. Now let us prove the inclusion $F^\times(1+P_E^{m-\mu}) \subset a^{-1}Ha \cap E^\times$. It is enough to take $\alpha \in (1+P_E^{m-\mu})$ and to show that $a\alpha = \gamma a$ with $\gamma \in H$ as before. If $\mu = 0$, then $(1+P_E^m) \subset H$ and $a \in K$; since K normalizes H, this case is clear. We need consider only $0 < \mu < m$. For the proof we intend to reverse the above reasoning. Certainly, we have $v_E(\alpha) = 0$ and, moreover, $v_E({}^{\sigma^i}\alpha - \alpha) \ge m-\mu$. From this we see that

$$({}^{\sigma^i}\alpha - \alpha)\beta_i \in P_E^m \qquad (i = 1,2). \tag{2.12}$$

Replacing γ_0 by $\alpha - (\gamma_1{}^{\sigma}\beta_2 + \gamma_2{}^{\sigma^2}\beta_1)$ in the last two equations of (2.11) produces the system of equations

$$({}^{\sigma}\alpha - \alpha)\beta_1 = (1-\beta_1{}^{\sigma}\beta_2)\gamma_1 + ({}^{\sigma^2}\beta_2 - {}^{\sigma^2}\beta_1\beta_1)\gamma_2,$$
$$({}^{\sigma^2}\alpha - \alpha)\beta_2 = ({}^{\sigma}\beta_1 - {}^{\sigma}\beta_2\beta_2)\gamma_1 + (1-{}^{\sigma^2}\beta_1\beta_2)\gamma_2.$$

Since $\mu > 0$, it follows that the determinant of the coefficient matrix of this system of two equations in the variables γ_1, γ_2 belongs to $\mathcal{O}_E^\times$. From (2.12) it follows that $\gamma_1, \gamma_2 \in P_E^m$; obviously, $\gamma_0 \in \mathcal{O}_E^\times$, given by the first equation of (2.11), satisfies $\gamma_0^{-1}\gamma_i \in P_E^m$. □

Our next task is to compute ${}^a\rho$ for $a \in H\backslash B/E^\times$. The above lemma tells us that ${}^a\rho \in (F^\times(1+P_E^{m-\mu}))\hat{}$ if $a \in \tilde{I}_{\mu,i}$. If $a' = a\xi^j$, then $a'^{-1}Ha' \cap E^\times = a^{-1}Ha \cap E^\times$ and ${}^{a'}\rho = {}^a\rho \circ \sigma^j$. Therefore it suffices to consider ${}^a\rho$ for $a \in \tilde{I}_{\mu,i}$. We need the following explicit form of $(1+\alpha_1\xi+\alpha_2\xi^2)^{-1}$.

Lemma 2.5. *Let $a = 1+\alpha_1\xi+\alpha_2\xi^2$ for $\alpha_1, \alpha_2 \in E$. Then*

$$a^{-1} = \frac{1}{\det(a)}((1-{}^\sigma\alpha_1{}^{\sigma^2}\alpha_2) + (\alpha_2{}^{\sigma^2}\alpha_2 - \alpha_1)\xi + (\alpha_1{}^\sigma\alpha_1 - \alpha_2)\xi^2),$$

where

$$\det(a) = 1 + n_E(\alpha_1) + n_E(\alpha_2) - \mathrm{tr}_E(\alpha_1{}^\sigma\alpha_2).$$

Proof. Put $(\beta_0+\beta_1\xi+\beta_2\xi^2)(1+\alpha_1\xi+\alpha_2\xi^2) = 1$ for $\beta_i \in E$. Since $\xi x\xi^{-1} = {}^\sigma x$ for $x \in E$, we have

$$\begin{pmatrix} 1 & {}^\sigma\alpha_2 & {}^{\sigma^2}\alpha_1 \\ \alpha_1 & 1 & {}^{\sigma^2}\alpha_2 \\ \alpha_2 & {}^\sigma\alpha_1 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the lemma follows from Cramer's rule. □

Lemma 2.6. *Let $c \in F^\times, y \in P_E^{m-\mu}$, $a = 1+\beta_1\xi+\beta_2\xi^2 \in \tilde{I}_{\mu,i}$ and $\gamma = \gamma_\theta \in E$ as in* (1.1). *Then*

$$\begin{aligned} &{}^a\rho\rho^{-1}(c(1+y)) = \psi_E(R_{\mu,1}(\varphi_2(\beta_1,\beta_2))({}^{\sigma^2}y - y)) &\quad (2.13)\\ &\qquad = \psi_E(R_{\mu,2}(\varphi_1(\beta_1,\beta_2))({}^\sigma y - y)), &\quad (2.14)\end{aligned}$$

where

$$R_{\mu,1}(u,v) = \frac{({}^{\sigma^2}\gamma - \gamma)u + \gamma v - {}^{\sigma^2}\gamma v^{-1}n_E(u)}{1+v+v^{-1}n_E(u) - \mathrm{tr}_E(u)},$$
$$R_{\mu,2}(v,u) = \frac{({}^\sigma\gamma - \gamma)u + \gamma v - {}^\sigma\gamma v^{-1}n_E(u)}{1+v+v^{-1}n_E(u) - \mathrm{tr}_E(u)},$$

and φ_i is as in Lemma 2.3 (2) *for $i = 1, 2$.*

Proof. Putting $g = 1+y$, we have

$$\begin{aligned} g^{-1}aga^{-1} &= 1 + (g^{-1}(a-1)g - (a-1))a^{-1} \\ &= 1 + (g^{-1}\beta_1{}^\sigma g\xi + g^{-1}\beta_2{}^{\sigma^2}g\xi^2 - (\beta_1\xi+\beta_2\xi^2))a^{-1} \\ &= 1 + (({}^\sigma gg^{-1}-1)\beta_1\xi + ({}^{\sigma^2}gg^{-1}-1)\beta_2\xi^2)a^{-1}. \end{aligned}$$

Using Lemma 2.5, we see that $\det(a)(g^{-1}aga^{-1}-1)$ equals

$$\begin{aligned} &(({}^\sigma gg^{-1}-1)\beta_1\xi + ({}^{\sigma^2}gg^{-1}-1)\beta_2\xi^2) \\ &\quad \times ((1-{}^\sigma\beta_1{}^{\sigma^2}\beta_2) + (\beta_2{}^{\sigma^2}\beta_2 - \beta_1)\xi + (\beta_1{}^\sigma\beta_1 - \beta_2)\xi^2). \end{aligned}$$

Since ${}^{\sigma^j}gg^{-1} - 1 \equiv {}^{\sigma^j}y - y \mod P_E^{2(m-\mu)}$, $\beta_i \in P_E^\mu$, $\rho|_{K^m} = \psi_\gamma$, $\mathrm{tr}_E(x\xi^i) = 0$ for $i = 1, 2$ and $x \in E$, we get

$$\begin{aligned}&{}^a\rho\rho^{-1}(1+y)\\ &= \psi_\gamma\Big(\frac{1}{\det(a)}(\beta_1({}^{\sigma}\beta_1{}^{\sigma^2}\beta_1 - {}^{\sigma}\beta_2)({}^{\sigma}y - y) + \beta_2({}^{\sigma^2}\beta_2{}^{\sigma^4}\beta_2 - {}^{\sigma^2}\beta_1)({}^{\sigma^2}y - y))\Big)\\ &= \psi_\gamma\left(\frac{1}{\det(a)}\left((n_E(\beta_1) - \beta_1{}^{\sigma}\beta_2)({}^{\sigma}y - y) + (n_E(\beta_2) - \beta_2{}^{\sigma^2}\beta_1)({}^{\sigma^2}y - y)\right)\right).\end{aligned}$$

Since $\mathrm{tr}_E\, u^{\sigma^j}v = \mathrm{tr}_E\, {}^{\sigma^{-j}}uv$ for any $u, v \in E$, we have

$$\mathrm{tr}_E(\gamma(n_E(\beta_1) - \beta_1{}^{\sigma}\beta_2)({}^{\sigma}y - y)) = \mathrm{tr}_E({}^{\sigma^2}\gamma(n_E(\beta_1) - {}^{\sigma^2}\beta_1\beta_2)(y - {}^{\sigma^2}y)).$$

Therefore we get

$${}^a\rho\rho^{-1}(1+y) = \psi_E\left(\frac{({}^{\sigma^2}\gamma - \gamma){}^{\sigma^2}\beta_1\beta_2 + \gamma n_E(\beta_2) - {}^{\sigma^2}\gamma n_E(\beta_1)}{\det(a)}({}^{\sigma^2}y - y)\right).$$

From $\det(a) = 1 + n_E(\beta_1) + n_E(\beta_2) - \mathrm{tr}_E(\beta_1{}^{\sigma}\beta_2)$, we obtain (2.14). The equation (2.13) can be shown in the same way. □

The next lemma is crucial for the character formula of $\kappa = \kappa_\theta$. We recall that $m = [(n+2)/2]$.

Lemma 2.7. *Fix $\alpha \in \varpi_F^\mu n_E(\mathcal{O}_E^\times)$ and put $\tilde{R}_{\mu,i}(x) = R_{\mu,i}(x, \alpha)$ for $x \in P_E^{2\mu+i-1}$. For $\mu = 0$, we assume $1 + \alpha + \alpha^{-1}x^3 - 3x \notin P_E$.*

1. If $\mu \geq 1$, then $\tilde{R}_{\mu,i}$ induces a bijection from $P_E^{2\mu+i-1}/P_E^{m+\mu}$ to $P_E^{2\mu+i-1-n}/P_E^{m+\mu-n}$.

2. $\tilde{R}_{0,2}$ induces a bijection from P_E/P_E^m to

$$\Big\{x \in P_E^{-n}/P_E^{m-n} \;\Big|\; x \equiv \frac{\gamma\alpha}{1+\alpha} \mod P_E^{1-n}\Big\}.$$

3. For any $x_0 \in \mathcal{O}_E$ such that $1 + \alpha + \alpha^{-1}x_0^3 - 3x_0 \notin P_E$, $\tilde{R}_{0,1}$ induces a bijection from $\{x \in \mathcal{O}_E/P_E^m | x \equiv x_0 \mod P_E\}$ to

$$\Big\{x \in P_E^{-n}/P_E^{m-n} \;\Big|\; x \equiv \frac{({}^{\sigma^2}\gamma - \gamma)x_0 + \gamma\alpha - {}^{\sigma^2}\gamma\alpha^{-1}x_0^3}{1 + \alpha + \alpha^{-1}x_0^3 - 3x_0} \mod P_E^{1-n}\Big\}.$$

Proof. First we assume $\mu > 0$. Let $x \in P_E^{2\mu+i-1}$. Then

$$\tilde{R}_{\mu,i}(x) \equiv ({}^{\sigma^{-i}}\gamma - \gamma)x \mod P_E^{v_E(x)-n+1}. \tag{2.15}$$

We remark that $v_E({}^{\sigma^{-i}}\gamma - \gamma) = -n$, since $n \not\equiv 0 \mod 3$. It follows from (2.15) that

$$\tilde{R}_{\mu,i}(x_1) \equiv \tilde{R}_{\mu,i}(x_2) + ({}^{\sigma^{-i}}\gamma - \gamma)(x_1 - x_2) \mod P_E^{v_E(x_1)+v_E(x_2-x_1)+i-n} \tag{2.16}$$

if $2\mu + i - 1 \leq v_E(x_1) \leq v_E(x_2)$. This implies $x_1 - x_2 \in P_E^m$ if $\tilde{R}_{\mu,i}(x_1) - \tilde{R}_{\mu,i}(x_2) \in P_E^{m-n}$. Thus the induced map from $P_E^{2\mu+i-1}/P_E^{m+\mu}$ to $P_E^{2\mu+i-1-n}/P_E^{m+\mu-n}$ is injective. It becomes automatically bijective.

Next we consider the case $\mu = 0$. For $x \in P_E$,

$$\tilde{R}_{0,2}(x) \equiv \frac{\gamma\alpha + ({}^{\sigma}\gamma - \gamma)x}{1+\alpha} \mod P_E^{v_E(x)+1-n}.$$

This implies the bijectivity of the induced map from $\tilde{R}_{0,2}$ by the same argument as above.

For $x_0 \in \mathcal{O}_E^\times$ and $x_1 \in P_E$,

$$\begin{aligned}\tilde{R}_{0,1}(x_0+x_1) &= \frac{({}^{\sigma^2}\gamma-\gamma)(x_0+x_1)+\gamma\alpha-{}^{\sigma^2}\gamma\alpha^{-1}n_E(x_0)n_E(1+x_0^{-1}x_1)}{1+\alpha+\alpha^{-1}n_E(x_0)n_E(1+x_0^{-1}x_1)-\mathrm{tr}_E(x_0+x_1)} \\ &\equiv \frac{({}^{\sigma^2}\gamma-\gamma)x_0+\gamma\alpha-{}^{\sigma^2}\gamma\alpha^{-1}n_E(x_0)+({}^{\sigma^2}\gamma-\gamma)x_1}{1+\alpha+\alpha^{-1}n_E(x_0)-\mathrm{tr}_E(x_0)} \mod P_E^{2-n} \\ &\equiv \tilde{R}_{0,1}(x_0)+\frac{({}^{\sigma^2}\gamma-\gamma)x_1}{1+\alpha+\alpha^{-1}n_E(x_0)-\mathrm{tr}_E(x_0)} \mod P_E^{2-n},\end{aligned}$$

since $n_E(1+x_0^{-1}x_1)$ and $\mathrm{tr}_E(x_1) \in P_F \subset P_E^3$. Thus the last assertion also follows in the same way as the proof for the case $\mu > 0$. □

Now we can describe the image of the map $a \mapsto {}^a\rho\rho^{-1}$ for $a \in \tilde{I}_{\mu,i}$.

Lemma 2.8. *Let $U_i = F^\times(1+P_E^i)$ for $i > 0$ and $U_0 = F^\times\mathcal{O}_E^\times$. For $\mu \geq 0$ and $i \in \{1,2\}$, we put $C(\mu,i) = q^{[(m-\mu-1)/3]+[(m+\mu-n+2)/3]-[(2\mu+i+1-n)/3]}$.*

1. For $\mu > 0$,

$$\bigoplus_{a\in\tilde{I}_{\mu,i}} {}^a\rho\rho^{-1} = \frac{q-1}{3}C(\mu,i)|P_E^{\mu-m+1}\cap F/P_E^{m+\mu-n}\cap F| \bigoplus_{\chi\in(U(m-\mu)/U(n+2-2\mu-i))^\wedge} \chi.$$

2. For $\alpha \in n_E(\mathcal{O}_E^\times)$, put

$$\Lambda(\alpha) = \left\{\chi \in (U_m/U_{n+1})^\wedge \,\middle|\, \chi(1+y) = \psi_E\left(\frac{\gamma\alpha}{1+\alpha}({}^{\sigma^2}y-y)\right) \text{ for } y \in P_E^n\right\}.$$

Then

$$\bigoplus_{a\in\tilde{I}_{0,2}} {}^a\rho\rho^{-1} = C(0,2) \bigoplus_{\alpha\in n_E(\mathcal{O}_E^\times)/1+P_F} \bigoplus_{\chi\in\Lambda(\alpha)} \chi.$$

3. For $\alpha \in n_E(\mathcal{O}_E^\times)$ and $x_0 \in \mathcal{O}_E$ satisfying $1+\alpha+\alpha^{-1}x_0^3-3x_0 \notin P_E$, let $\Omega(\alpha,x_0)$ be a subset of $(U_m/U_{n+1})^\wedge$ consisting of characters χ such that

$$\chi(1+y) = \psi_E\left(\frac{({}^{\sigma^2}\gamma-\gamma)x_0+\gamma\alpha-{}^{\sigma^2}\gamma\alpha^{-1}x_0^3}{1+\alpha+\alpha^{-1}x_0^3}({}^{\sigma}y-y)\right) \quad \text{for } y \in P_E^n.$$

Then

$$\bigoplus_{a\in\tilde{I}_{0,1}} {}^a\rho\rho^{-1} = C(0,2) \bigoplus_{\alpha\in n_E(\mathcal{O}_E^\times)/1+P_F} \bigoplus_{x_0\in\mathcal{O}_E/P_E} \bigoplus_{\chi\in\Omega(\alpha,x_0)} \chi.$$

Proof. By Lemmas 2.7 and 2.8, it suffices to calculate the multiplicity of the character χ in the right-hand side. The map

$$\begin{aligned} P_E^{2\mu-i-1-n}/P_E^{m+\mu-n} &\to (U_{m-\mu}/U_{n+1-2\mu+1-i})^\wedge, \\ a &\mapsto (c(1+y) \mapsto \psi_E(a({}^\sigma y-y))), \end{aligned}$$

is surjective and every fiber of the map has

$$|P_E^{2\mu-i-1-n}\cap F/P_E^{m+\mu-n}\cap F||P_E^{\mu-m+1}\cap F/P_E^{m+\mu-n}\cap F|$$

elements. Since $P_E^j\cap F = P_F^{[(j+2)/3]}$, $|n_E(\mathcal{O}_E^\times)/1+P_F^{[(m-\mu+2)/3]}| = \frac{q-1}{3}q^{[(m-\mu-1)/3]}$,

$$m+\mu-n = \begin{cases} \mu-m+1 & \text{if} \quad n = 2m-1, \\ \mu-m+2 & \text{if} \quad n = 2m-2, \end{cases}$$

and $m \not\equiv 1 \mod 3$ when $n = 2m-2$, we get this lemma from Lemma 2.6 and Lemma 2.7. □

Now we can give the character formula of $\kappa = \kappa_\theta$ on $E^\times$.

Proposition 2.9. *Put $C = q^{[(m-\mu-1)/3]+[(m+\mu-n+2)/3]-[(2\mu+i+1-n)/3]}$ for $\mu \geq 0$ and $i \in \{1,2\}$.*

(1) (*Decomposition of κ as $E^\times$-module*) *The restriction of κ to $E^\times$ is decomposed as follows:*

$$\left(\bigoplus_{i=0}^{2} \theta \circ \sigma^i\right) \otimes \left(\{1\} + \sum_{j=1}^{n-1} \frac{q-1}{3} q^{c_j} X_j\right)$$
$$\oplus\, C \left(\theta \otimes \bigoplus_{\alpha \in n_E(\mathcal{O}_E^\times)/1+P_F} \bigoplus_{\chi \in \tilde{\Lambda}(\alpha)} \chi \right.$$
$$\oplus\, (\theta \circ \sigma^2) \otimes \bigoplus_{\alpha \in n_E(\mathcal{O}_E^\times)/1+P_F} \bigoplus_{\chi \in \tilde{\Lambda}(\alpha)} \chi$$
$$\left. \oplus\, \theta \otimes \bigoplus_{\alpha \in n_E(\mathcal{O}_E^\times)/1+P_F} \bigoplus_{x_0 \in \mathcal{O}_E/P_E} \bigoplus_{\chi \in \tilde{\Omega}(\alpha, x_0)} \chi \right),$$

where $X_j = \bigoplus_{\chi \in E^\times/U_j} \chi$, $c_j = 2[\frac{j+2}{6}]$,

$$\tilde{\Lambda}(\alpha) = \left\{ \chi \in (E^\times/U_{n+1})\hat{}\ \middle|\ \chi(1+y) = \psi_E\left(\frac{\gamma\alpha}{1+\alpha}({}^{\sigma^2}y - y)\right) \text{ for } y \in P_E^n \right\}$$

and $\Omega(\alpha, x_0)$ is a subset of $(E/U_{n+1})\hat{}$ consisting of characters χ such that

$$\chi(1+y) = \psi_E\left(\frac{({}^{\sigma^2}\gamma - \gamma)x_0 + \gamma\alpha - {}^{\sigma^2}\gamma\alpha^{-1}x_0^3}{1+\alpha+\alpha^{-1}x_0^3}({}^{\sigma}y - y)\right) \text{ for } y \in P_E^n.$$

(2) (*Character formula of κ on $E^\times$*) *The character χ_κ of κ is given by*

x	$\chi_\kappa(x)$
$x \in E^\times - U_0$	$\sum_{i=0}^{2} \theta({}^{\sigma^i}x)$
$x \in U_j^* \quad (1 \leq j \leq n-1)$	$q^j \sum_{i=0}^{2} \theta({}^{\sigma^i}x)$
$c(1+y) \in U_n^*\ (c \in F^\times, y \in \varpi_E^n \mathcal{O}_E^\times)$	$q^{n-1}\theta(c)\,\mathrm{Kl}(\gamma y)$
$x \in U_{n+1}$	$q^{n-1}(q-1)^2\theta(x)$

where the Kloosterman sum $\mathrm{Kl}(a)$ is defined by

$$\mathrm{Kl}(a) = \sum_{\substack{(y_0,y_1,y_2) \in k_F^3 \\ y_0y_1y_2 = a}} \psi(\gamma\varpi_E^n(y_0+y_1+y_2)). \tag{2.17}$$

(Since $\gamma\varpi_E^n \in \mathcal{O}_E$ and $k_E = k_F$, we regard $\gamma\varpi_E^n \mod P_E$ as an element of k_F.)

Proof. If $n = 2m-1$, then it follows from Lemma 2.8 that

$$\bigoplus_{a\in\tilde{I}_{\mu,i}} \operatorname{Ind}_{U_{m-\mu}}^{E^\times} {}^a\rho\rho^{-1}$$
$$= \begin{cases} \dfrac{q-1}{3}C(\mu,i)X_{n+1-2\mu-i+1} & \text{for } \quad \mu > 0, \\ C(0,2) \displaystyle\bigoplus_{\alpha\in n_E(\mathcal{O}_E^\times)/1+P_F^{[(m+2)/3]}} \bigoplus_{\chi\in\tilde{\Lambda}(\alpha)} \chi & \text{for } \mu = 0 \text{ and } i = 2, \\ C(0,2) \displaystyle\bigoplus_{\substack{\alpha\in n_E(\mathcal{O}_E^\times)/1+P_F^{[(m+2)/3]} \\ x_0\in k_F}} \bigoplus_{\chi\in\tilde{\Omega}(\alpha,x_0)} \chi & \text{for } \mu = 0 \ \text{ and } i = 1. \end{cases}$$

By virtue of (2.7) and (2.3), we can obtain the decomposition of π as an $E^\times$-module. When $n = 2m-2$, we get by Lemma 2.8 that

$$\bigoplus_{a\in\tilde{I}_{\mu,i}} \operatorname{Ind}_{U_{m-\mu}}^{E^\times} {}^a\rho\rho^{-1}$$
$$= \begin{cases} \dfrac{q-1}{3}C(\mu,i)X_{n+1-2\mu-i+1} & \text{for } \mu > 0 \text{ and } \mu \equiv m-1 \mod 3, \\ \dfrac{q-1}{3}qC(\mu,i)X_{n+1-2\mu-i+1} & \text{for } \mu > 0 \text{ and } \mu \not\equiv m-1 \mod 3, \\ C(0,2) \displaystyle\bigoplus_{\alpha\in n_E(\mathcal{O}_E^\times)/1+P_F^{[(m+2)/3]}} \bigoplus_{\chi\in\tilde{\Lambda}(\alpha)} \chi & \text{for } \mu = 0 \text{ and } i = 2, \\ C(0,2) \displaystyle\bigoplus_{\substack{\alpha\in n_E(\mathcal{O}_E^\times)/1+P_F^{[(m+2)/3]} \\ x_0\in k_F}} \bigoplus_{\chi\in\tilde{\Omega}(\alpha,x_0)} \chi & \text{for } \mu = 0 \text{ and } i = 1. \end{cases}$$

By (2.8) and (2.3), we get the first part of the theorem. Except on U_n^*, we can easily obtain the character formula of κ from the decomposition of κ as an $E^\times$-module, Lemma 2.8 and the fact that $|E^\times/U_j| = 3q^{[2j/3]}$. For $x \in U_n^*$, we can prove it by the decomposition of κ and Lemma 2.6. But the calculation of the character on K^n can be obtained directly from the definition of the induced representation. We state it, including the case $x \notin E$.

Lemma 2.10. *Let $x = 1 + \varpi_E^n x_0$ for $x_0 = \operatorname{diag}(k_1,k_2,k_3)$ $(k_i \in \mathcal{O}_F^\times)$ and the Kloosterman sum $\operatorname{Kl}(a)$ as in* (2.17). *Then*

$$\chi_\kappa(x) = q^{n-1}\operatorname{Kl}(k_1k_2k_3).$$

Proof. By the definition of κ and Frobenius's formula, we have

$$\chi_\kappa(1+\varpi_E^n\operatorname{diag}(k_1,k_2,k_3)) = q^{n-1}\sum_{a\in E^\times K^1\backslash B}\psi(\operatorname{Tr}\gamma a\varpi_E^n\operatorname{diag}(k_1,k_2,k_3)a^{-1}).$$

It follows from (2.3) and (2.5) that the set $\{\operatorname{diag}(1,y,z) | y,z\in k_F^\times\}$ becomes a complete set of representatives of $E^\times K^1\backslash B$. Since $\varpi_E\operatorname{diag}(1,y,z)\varpi_E^{-1} = \operatorname{diag}(y,z,1)$, we have

$$\operatorname{Tr}\gamma\operatorname{diag}(1,y,z)\varpi_E^n\operatorname{diag}(k_1,k_2,k_3)\operatorname{diag}(1,y,z)^{-1} \mod P_F$$
$$= \begin{cases} \gamma\varpi_E^n(zk_1+y^{-1}k_2+yz^{-1}k_3) \mod P_F & \text{if } n\equiv 1 \mod 3, \\ \gamma\varpi_E^n(yk_1+zy^{-1}k_2+z^{-1}k_3) \mod P_F & \text{if } n\equiv 2 \mod 3. \end{cases}$$

Hence our lemma. □

By applying the above lemma for $\operatorname{diag}(k_1, k_2, k_3) = \operatorname{diag}(x_0, x_0, x_0)$, we get the character formula of κ on U_n^*. $\square$

The character outside of the conjugacy class of $E^\times$ can be obtained easily.

Lemma 2.11. *Let x be a regular elliptic element of B. If x satisfies the conditions that $F(x) \not\simeq E$ and x is not conjugate to an element of $F^\times K^n$, then $\chi_\kappa(x) = 0$.*

Proof. See Lemma 3.3 in [15]. $\square$

We remark that if $F(x)|F$ is unramified, then $F(x) \cap K^n = F(x) \cap K^{n+1}$ since $n \not\equiv 0 \mod 3$. Let $E'|F$ be a ramified extension in $\mathrm{M}_3(F)$. Then either $E' \sim E$ or $E' \sim F(x)$, where $x = \varpi_E \operatorname{diag}(b, 1, 1)$ with $b \in \mathcal{O}_F^\times$ not a cube mod P_F. The element x is obviously a prime element of $\mathcal{O}_{F(x)}$. If $g \in F(x)$ and $v_{F(x)}(g-1) = n$, then $g = 1 + x^n y + z$ for some $y \in k_F^\times$ and $z \in P_{F(x)}^{n+1}$.

Proposition 2.12. *Let x be a regular elliptic element of B such that $F(x) \not\simeq E$.*

1. If $F(x)|F$ is unramified, χ_κ is given by

x	$\chi_\kappa(x)$
$x \notin F^\times(K^{n+1} \cap F(x)))$	0
$x = cy \ (c \in F^\times, y \in K^{n+1} \cap F(x))$	$q^{n-1}(q-1)^2$

2. If $F(x)|F$ is ramified, then χ_κ is given by

x	$\chi_\kappa(x)$
$x \notin F^\times(1 + P_{F(x)}^n)$	0
$c(1 + \varpi_{F(x)}^n \operatorname{diag}(k_1, k_2, k_3) + z)$ $(c \in F^\times, k_i \in k_F^\times, z \in P_{F(x)}^{n+1})$	$q^{n-1}\theta(c)\,\mathrm{Kl}(b^{3[(n+4)/3]} k_1 k_2 k_3)$
$c(1+y)$ *for* $c \in F^\times, y \in P_{F(x)}^{n+1}$	$q^{n-1}(q-1)^2\theta(c)$

Proof. This follows from Lemma 2.11, Lemma 2.10 and the remark above this proposition. $\square$

As pointed out in Theorem 1.6, most of the character formula for $\pi = \pi_\theta$ (on the regular elliptic set) comes from the formula for $\kappa = \kappa_\theta$.

Theorem 2.13. *Let E be a cubic ramified extension of F, let θ be a generic quasi-character of $E^\times$ such that $f(\theta) = n+1$, and let π_θ be the corresponding irreducible supercuspidal representation of $\mathrm{GL}_3(F)$ (see Theorem 1.3). Put $U_0 = F^\times \mathcal{O}_E^\times$, $U_j = F^\times(1 + P_E^j)$ and $U_j^* = U_j - U_{j+1}$ for $j \geq 1$. Let ξ be an element of $\mathrm{GL}_3(F)$ such that $\xi^3 = 1$ and ${}^\sigma x = \xi x \xi^{-1}$ for $x \in E$, where $\langle\sigma\rangle = \mathrm{Gal}(E|F)$. Let x be a regular elliptic element of $\mathrm{GL}_3(F)$.*

1. If $F(x)|F$ is unramified, then

x	$\chi_{\pi_\theta}(x)$
$x \notin F^\times(1 + P_{F(x)}^{n+1})$	0
$c(1+y)(c \in F^\times, y \in P_{F(x)}^{n+1})$	$q^{n-1}(q^2+q+1)\theta(c)$

2. If $x \in E$, then

x	$\chi_{\pi_\theta}(x)$
$x \in E^\times - U_0$	$\sum_{i=0}^{2} \theta({}^{\sigma^i}x)$
$x \in U_j^* \quad (1 \le j \le n-1)$	$q^j \sum_{i=0}^{2} \theta({}^{\sigma^i}x)$
$c(1+y) \in U_n^*\ (c \in F^\times, y \in \varpi_E^n \mathcal{O}_E^\times)$	$q^{n-1}\theta(c)\,\mathrm{Kl}(\gamma y)$
$x \in U_{n+1}$	$q^{n-1}(q^2+q+1)\theta(x)$

where the Kloosterman sum $\mathrm{Kl}(a)$ *is defined by* (2.17).

3. If $F(x)|F$ is ramified and $F(x) \not\simeq E$, let b be as in the remark above this proposition and put $\varpi_{F(x)} = \varpi_E \operatorname{diag}(b,1,1)$. Then

x	$\chi_{\pi_\theta}(x)$
$x \notin F^\times(1+P_{F(x)}^n)$	0
$c(1+\varpi_{F(x)}^n \operatorname{diag}(k_1,k_2,k_3)+z)$ $(c \in F^\times, k_i \in k_F^\times, z \in P_{F(x)}^{n+1})$	$q^{n-1}\theta(c)\,\mathrm{Kl}(b^{3[(n+4)/3]}k_1k_2k_3)$
$c(1+y)$ *for* $c \in F^\times, y \in P_{F(x)}^{n+1}$	$q^{n-1}(q^2+q+1)\theta(c)$

Proof. Except on $K^{n+1} \cap F(x)$ the formula of this theorem follows from Theorem 1.8 and the character formula for κ, given in Propositions 2.9 and 2.12. To complete the formula by determining it on $K^{n+1} \cap F(x)$, we use the abstract matching theorem ([16], [8] and [2]) and the fact, which we have already remarked, that in our case the Howe-Moy correspondence respects abstract matching. The representation π'_θ of $D^\times$ which corresponds to the generic character θ has dimension $q^{n-1}(q^2+q+1)$ and is trivial on $1+P_D^{n+1}$. Therefore, by Proposition 1.5, $\chi_{\pi_\theta}(x) = q^{n-1}(q^2+q+1)$ for all regular elliptic $x \in K_{n+1}$. □

3. Character formula for the non-Galois case

In this section we treat the case in which F does not contain cube roots of unity, i.e., the case in which $q = p^f \equiv 2 \bmod 3$. In this case the extension $E|F$ is not Galois; so we cannot use the methods of the previous section. In order to apply the results obtained in the Galois case we use the base change lift of $\kappa_\theta|_{B^1}$. Let L be an unramified quadratic extension of F. Then $L = F(\zeta)$, where ζ is, as before, a primitive cube root of unity. We write τ for a generator of $\mathrm{Gal}(L|F)$ (${}^\tau\zeta = \zeta^2$) and add the subscript "L" to objects which are extended by L, e.g. $E_L = E \otimes_F L \cong EL$. The field extension $E_L|L$ is ramified and Galois, $E_L|E$ is a quadratic unramified extension, and $\langle\tau\rangle \cong \mathrm{Gal}(E_L|E) \cong \mathrm{Gal}(L|F)$. Thus $E_L|F$ is an $\mathfrak{S}_3$ extension. We embed $E \hookrightarrow E_L$ by sending $x \mapsto x \otimes 1$ and, as in the previous section, we identify $\mathrm{M}_3(L) = \mathrm{M}_3(F) \otimes_F L$ with $\mathrm{End}_L(E_L)$. Similarly, we set $G_L = \mathrm{GL}_3(L)$ and use the L-basis $\{\varpi_E^2, \varpi_E, 1\}$ of E_L to identify G_L with $\mathrm{Aut}_L(E_L)$; we note that this basis is also an $\mathcal{O}_L$-basis for $\mathcal{O}_{E_L}$. Using the lattice flag $\{P_{E_L}^i\}_{i\in\mathbb{Z}}$, we define

$$A_L^i = \{f \in \mathrm{M}_3(L) \mid f(P_{E_L}^j) \subset P_{E_L}^{j+i} \text{ for all } j \in \mathbb{Z}\}.$$

Put $K_L = (A_L^0)^\times, B_L = E_L^\times K_L$ and $K_L^i = 1 + A_L^i$ for $i \ge 1$. By Kutzko [15], it suffices to calculate the character of $\kappa = \kappa_\theta$ instead of π_θ. In fact, we have only to determine the character of $\kappa|_{B^1}$. Therefore we have only to treat the base change of $\kappa|_{B^1}$ to B_L^1, where $B_L^1 = L^\times K_L$.

Definition 3.1. Let θ be a generic character of $E^\times$ with $f(\theta) = n + 1$ and $\theta(1 + x) = \psi(\mathrm{tr}_E(\gamma x))$ for $x \in P_E^m$. We define a base change lift θ_L of θ to $L^\times$ by setting $\theta_L = \theta \circ n_{E_L|E}$. Then $\theta_L(1 + x) = \psi_L(\mathrm{tr}_{E_L|L}\,\gamma x)$ for $x \in P_{E_L}^m$. (Recall that $m = [(n + 2)/2]$.) The base change lift ρ_L of $\rho|_{H^1}$ to $H_L^1 = L^\times(1 + P_{E_L})K_L^m$ is defined by

$$\rho_L(h \cdot g) = \theta_L(h)\psi_L(\mathrm{Tr}\,\gamma(g - 1)) \qquad \text{for} \quad h \in L^\times(1 + P_{E_L}), \quad g \in K_L^m.$$

When $n + 1 = 2m$, we define the base change κ_L of $\kappa|_{B^1}$ to B_L^1 by

$$\kappa_L = \mathrm{Ind}_{H_L^1}^{B_L^1} \rho_L.$$

When $n + 1 = 2m - 1$, there exists an irreducible representation η_L of $J_L^1 = L^\times(1 + P_{E_L})K_L^{m-1}$ such that

$$\mathrm{Ind}_{H_L^1}^{J_L^1} \rho_L = q^2\eta_L.$$

Then the base change lift κ_L of $\kappa_\theta|_{B^1}$ to B_L^1 is given by

$$\kappa_L = \mathrm{Ind}_{J_L^1}^{B_L^1} \eta_L.$$

By virtue of $\theta_L \circ \tau = \theta_L$, we have $\rho_L \circ \tau = \rho_L$. Thus we can define an extension $\tilde{\rho}_L$ of ρ_L to $H_L^1 \rtimes \langle\tau\rangle$ by

$$\tilde{\rho}_L(x \rtimes \tau) = \rho_L(x) \quad \text{for } x \in H_L^1.$$

Similarly we can extend η_L to $J_L^1 \rtimes \langle\tau\rangle$ since $\eta_L \circ \tau \simeq \eta_L$.

Now we apply the result of Bushnell and Henniart ([4]) to our case and get the character relation of κ_L and $\tilde{\kappa}_L$.

Lemma 3.2. *There exists an extension $\tilde{\eta}_L$ of η_L to $J_L^1 \rtimes \langle\tau\rangle$ such that*

$$\chi_{\tilde{\eta}_L}(x \rtimes \tau) = \chi_\eta(n_{E_L|E}(x)) \quad \textit{for } x \in E_L^\times$$

Proof. Let $\tilde{\eta}$ be an extension of η_L to $J_L^1 \rtimes \langle\tau\rangle$. It follows from Proposition (12.8) in [4] that

$$\chi_{\tilde{\eta}}(x \rtimes \tau) = \frac{\chi_{\tilde{\eta}}(\tau)}{q}\chi_\eta(n_{E_L|E}(x)).$$

Applying Lemma (13.1) Proposition (ii) to our case, we have

$$|\chi_{\tilde{\eta}}(\tau)| = |{}^\tau(J_L^1/H_L^1)|^{1/2} = q^2.$$

Since τ is order 2, $\chi_{\tilde{\eta}}(\tau) = \overline{\chi_{\tilde{\eta}}}$. Thus we obtain $\chi_{\tilde{\eta}}(\tau) = \pm q$. If $\chi_{\tilde{\eta}}(\tau) = -q$, the other extension of η_L to $J_L^1 \rtimes \langle\tau\rangle$ satisfies the desired equation. □

Using Frobenius's Formula and Corollary (12.19) in [4], we see that the following result holds.

Proposition 3.3. *Let $x \in L^\times(1 + P_{E_L})$.*

1. *When $n + 1 = 2m$,*

$$\chi_\kappa(n_{E_L|E}(x)) = \sum_{\substack{a \in H_L^1\backslash B_L^1 \\ ax^\tau a^{-1} \in H_L^1}} \rho_L(ax^\tau a^{-1}). \tag{3.1}$$

2. When $n+1=2m-1$,

$$\begin{aligned}\chi_\kappa(n_{E_L|E}(x)) &= \sum_{\substack{a\in J_L^1\backslash B_L^1\\ ax\,{}^\tau a^{-1}\in J_L^1}} \chi_{\eta_L}(ax{}^\tau a^{-1}\rtimes\tau)\\ &= q\sum_{\substack{a\in J_L^1\backslash B_L^1\\ ax\,{}^\tau a^{-1}\in H_L^1}} \rho_L(ax{}^\tau a^{-1}).\end{aligned} \tag{3.2}$$

We recall that $J_L^1=L^\times(1+P_{E_L})K_L^{m'}$ for $m'=[(n+1)/2]$. Since $m'=m$ if $n+1=2m$, we get $J_L^1=H_L^1$ if $n+1=2m$. Thus we have only to consider the coset $J_L^1\backslash B_L^1$.

We proceed in the same way as section 2. Set $\xi=\mathrm{diag}(1,\zeta^2,\zeta)\in \mathrm{M}_3(L)$. Then ξ satisfies $\xi^3=1$, ${}^\tau\xi=\xi^2$,

$$\xi x\xi^{-1}={}^\sigma x \quad \text{for any } x\in E_L$$

and

$$\begin{array}{rcccccc}\mathrm{M}_3(L) &=& E_L &\oplus& E_L\xi &\oplus& E_L\xi^2,\\ A_L^0 &=& \mathcal{O}_{E_L} &\oplus& \mathcal{O}_{E_L}\xi &\oplus& \mathcal{O}_{E_L}\xi^2,\\ A_L^1 &=& P_{E_L} &\oplus& P_{E_L}\xi &\oplus& P_{E_L}\xi^2,\\ A_L^2 &=& P_{E_L}^2 &\oplus& P_{E_L}^2\xi &\oplus& P_{E_L}^2\xi^2.\end{array} \tag{3.3}$$

By Lemma 2.2, the set

$$\begin{aligned}&\{1+\beta_1\xi+\beta_2\xi^2|\beta_i\in\mathcal{O}_{E_L}/P_{E_L}^{m'},\det(1+\beta_1\xi+\beta_2\xi^2)\in\mathcal{O}_{E_L}^\times\}\\ &\cup\{(1+\beta_1\xi+\beta_2\xi^2)\xi|\beta_1\in P_{E_L}/P_{E_L}^{m'},\beta_2\in P_{E_L}/P_{E_L}^{m'}\}\\ &\cup\{(1+\beta_1\xi+\beta_2\xi^2)\xi^2|\beta_1\in P_{E_L}/P_{E_L}^{m'},\beta_2\in \mathcal{O}_{E_L}/P_{E_L}^{m'},\det(1+\beta_2\xi^2)\in\mathcal{O}_{E_L}^\times\}\end{aligned} \tag{3.4}$$

gives a complete set of representatives of $J_L\backslash B_L$. If $y\in U_i^*$, then there exists an element $x\in L^\times(1+P_{E_L}^i)-L^\times(1+(P_{E_L}^i\cap \mathrm{Ker}\,\mathrm{tr}_{E_L|E})+P_{E_L}^{i+1})$ such that $n_{E_L|E}(x)=y$. Thus it suffices to calculate the right-hand sides of (3.1) and (3.2) for $x\in L^\times(1+P_{E_L}^i)-L^\times(1+(P_{E_L}^i\cap \mathrm{Ker}\,\mathrm{tr}_{E_L|E})+P_{E_L}^{i+1})$ with $i\geq 0$. The case $n_{E_L|E}(x)\in E^\times-U_1$ is treated in Lemma 3.8. Now we assume $i\geq 1$.

Lemma 3.4. *Let* $x\in L^\times(1+P_{E_L}^i)-L^\times(1+(P_{E_L}^i\cap \mathrm{Ker}\,\mathrm{tr}_{E_L|E})+P_{E_L}^{i+1})$ *for* $i\geq 1$ *and* $a\in B_L$.

1. $ax\,{}^\tau a^{-1}\in H_L$ *if and only if* $H_La=H_L\,{}^\tau a$ *and* $H_La=H_Lax$.

2. When $a=1+\beta_1\xi+\beta_2\xi^2$ *for* $\beta_1,\beta_2\in\mathcal{O}_{E_L}$, $ax\,{}^\tau a^{-1}\in H_L$ *is equivalent to* $\beta_1,\beta_2\in P_{E_L}^{m'-i}$ *and* $\beta_2={}^\tau\beta_1 \bmod P_{E_L}^{m'}$.

3. When $a=(1+\beta_1\xi+\beta_2\xi^2)\xi^2$ *for* $\beta_1\in P_{E_L}$ *and* $\beta_2\in\mathcal{O}_{E_L}$, $ax\,{}^\tau a^{-1}\in H_L$ *is equivalent to* $i\geq m'$, $n_{E_L|E}(\beta_2)\in 1+P_E^{m'}$ *and* $\beta_1=({}^\tau\beta_2)^{-1}\,{}^\tau\beta_1 \bmod P_{E_L}^{m'}$.

4. When $a=(1+\beta_1\xi+\beta_2\xi^2)\xi$ *for* $\beta_1,\beta_2\in P_{E_L}$, $ax\,{}^\tau a^{-1}\notin H_L$.

Proof. First we prove 2. If $ax\,{}^\tau a^{-1}\in H_L$, then there exist $\gamma_0\in\mathcal{O}_{E_L}^\times$ and $\gamma_1,\gamma_2\in P_{E_L}^{m'}$ such that

$$(1+\beta_1\xi+\beta_2\xi^2)x=\gamma_0(1+\gamma_1\xi+\gamma_2\xi^2)(1+{}^\tau\beta_2\xi+{}^\tau\beta_1\xi^2).$$

Therefore,

$$\begin{aligned} x &= \gamma_0(1+\gamma_1\,{}^{\sigma\tau}\beta_1+\gamma_2\,{}^{\sigma^2\tau}\beta_2),\\ \beta_1\,{}^{\sigma}x &= \gamma_0(\gamma_1+{}^{\tau}\beta_2+\gamma_2\,{}^{\sigma^2\tau}\beta_1),\\ \beta_2\,{}^{\sigma^2}x &= \gamma_0(\gamma_2+{}^{\tau}\beta_1+\gamma_1\,{}^{\sigma\tau}\beta_2). \end{aligned}$$

By multiplying the first equation on the right successively by ${}^{\tau}\beta_2$ and ${}^{\tau}\beta_1$ and using the fact that $\gamma_1,\gamma_2\in P_{E_L}^{m'}$, we see that

$$\begin{aligned} \beta_1\,{}^{\sigma}x &= x\,{}^{\tau}\beta_2 \bmod P_{E_L}^{m'},\\ \beta_2\,{}^{\sigma^2}x &= x\,{}^{\tau}\beta_1 \bmod P_{E_L}^{m'}. \end{aligned}$$

Eliminating β_2 from these two equations and using the fact that $\tau\sigma^2=\sigma\tau$, we obtain

$$\beta_1 = n_{E_L|E}(x)\,{}^{\sigma}n_{E_L|E}(x)^{-1}\beta_1 \bmod P_{E_L}^{m'}.$$

Since $n_{E_L|E}(x)\in U_i^*$, we have $v_E(n_{E_L|E}(x)\,{}^{\sigma}n_{E_L|E}(x)^{-1}-1)=i$, and so $\beta_1\in P_{E_L}^{m'-i}$; similarly, $\beta_2\in P_{E_L}^{m'-i}$. Since $x\,{}^{\sigma}x^{-1}\in 1+P_{E_L}^{i}$, it follows that $\beta_1={}^{\tau}\beta_2$ mod $P_{E_L}^{m'}$. Then it follows from Lemma 2.2 that ${}^{\tau}a\equiv a$ mod H_L. Next we prove 3. If $ax\,{}^{\tau}a^{-1}\in H_L$, then there exist $\gamma_0\in\mathcal{O}_{E_L}^{\times}$ and $\gamma_1,\gamma_2\in P_{E_L}^{m'}$ such that

$$(\xi^2+\beta_1+\beta_2\xi)x=\gamma_0(1+\gamma_1\xi+\gamma_2\xi^2)(\xi+{}^{\tau}\beta_1+{}^{\tau}\beta_2\xi^2).$$

From ${}^{\tau}\xi=\xi^2$ and ${}^{\tau}\xi^2=\xi$, this implies that

$$\begin{aligned} {}^{\sigma^2}x &= \gamma_0(\gamma_1+\gamma_2{}^{\sigma^2\tau}\beta_1+{}^{\tau}\beta_2),\\ \beta_1x &= \gamma_0({}^{\tau}\beta_1+\gamma_1{}^{\sigma\tau}\beta_2+\gamma_2),\\ \beta_2{}^{\sigma}x &= \gamma_0(1+\gamma_1{}^{\sigma\tau}\beta_1+\gamma_2{}^{\sigma^2\tau}\beta_2). \end{aligned}$$

In the same way as the proof of 1, we can show that

$$(3.5)\qquad {}^{\sigma^2}x=n_{E_L|E}(\beta_2){}^{\sigma}x \mod P_{E_L}^{m'} \quad\text{and}\quad \beta_1x={}^{\sigma}x\beta_2{}^{\tau}\beta_1 \mod P_{E_L}^{m'}.$$

Since $n_{E_L|E}(\beta_2)$ is τ-invariant, we have ${}^{\tau\sigma}({}^{\sigma}xx^{-1})={}^{\sigma}({}^{\sigma}xx^{-1})$ mod $P_{E_L}^{m'}$. Using $\tau\sigma^2=\sigma\tau$ and $\tau\sigma=\sigma^2\tau$, we get ${}^{\sigma}({}^{\tau}xx)={}^{\sigma^2}({}^{\tau}xx)$ mod $P_{E_L}^{m'}$. This implies ${}^{\sigma}n_{E_L|E}(x)=n_{E_L|E}(x)$ mod $P_{E_L}^{m'}$. Therefore we obtain $n_{E_L|E}(x)\in F^{\times}(1+P_E^{m'})$. By the assumption $x\in L^{\times}(1+P_{E_L}^{i})-L^{\times}(1+(P_{E_L}^{i}\cap\operatorname{Ker}\operatorname{tr}_{E_L|E})+P_{E_L}^{i+1})$, we get $i\geq m'$ and ${}^{\sigma}xx^{-1}\in 1+P_{E_L}^{m'}$. It follows from (3.5) that $n_{E_L|E}(\beta_2)\in 1+P_{E_L}^{m'}$ and $\beta_1=({}^{\tau}\beta_2)^{-1}{}^{\tau}\beta_1$ mod $P_{E_L}^{m'}$. This implies $H_La=H_L{}^{\tau}a$ by Lemma 2.2. Finally we prove 4. By the same argument as above, $ax\,{}^{\tau}a^{-1}\in H_L$ implies $\beta_2x\equiv{}^{\sigma^2}x\beta_1{}^{\tau}\beta_2$ mod $P_{E_L}^{m'}$. Since $\beta_1\in P_{E_L}$, $v_{E_L}({}^{\sigma^2}x\beta_1{}^{\tau}\beta_2)\geq v_{E_L}(\beta_2)+1$. This is impossible. As for 1, it follows from (3.4) and the proofs of 2, 3 and 4. □

Lemma 3.5. 1. *Let $x\in\mathcal{O}_{E_L}^{\times}$ satisfy $n_{E_L|E}(x)\in U_i^*$ $(i>0)$ and $a=1+\beta\xi+{}^{\tau}\beta\xi^2$ for $v_{E_L}(\beta)\geq\max(0,m'-i)$. Then*

$$\rho_L(ax^{\tau}a^{-1}x^{-1})=\psi_E(S(\beta)\operatorname{tr}_{E_L|E}(x-1)),$$

where

$$(3.6)\qquad S(\beta)=\frac{\mathrm{tr}_{E_L|E}((\gamma-{}^{\sigma}\gamma)(n_{E_L|{}^{\sigma^2}E}(\beta)-n_{E_L|L}({}^{\tau}\beta)))}{1+\mathrm{tr}_{L|F}(n_{E_L|L}(\beta))-\mathrm{tr}_{{}^{\sigma^2}E|F}(n_{E_L|{}^{\sigma^2}E}(\beta))}.$$

2. *Let* $\beta_1\in P_{E_L}$, $\beta_2\in\mathcal{O}_{E_L}$ *satisfy* $n_{E_L|E}(\beta_2)=1$ *and* $\beta_1=({}^{\tau}\beta_2)^{-1}{}^{\tau}\beta_1$. *Put* $a=(1+\beta_1\xi+\beta_2\xi^2)\xi^2$. *For* $n_{E_L|E}(x)\in U_{m'}$,

$$\rho_L(ax^{\tau}a^{-1}x^{-1})=\psi_E(T(\beta_1,\beta_2)\,\mathrm{tr}_{E_L|E}(x-1)),$$

where

$$(3.7)\qquad T(\beta_1,\beta_2)=\frac{({}^{\sigma}\gamma-\gamma)(1-{}^{\sigma^2}\beta_1\beta_2)+({}^{\sigma^2}\gamma-\gamma)(n_{E_L|L}(\beta_2)-{}^{\sigma}\beta_1{}^{\sigma^2}\beta_2)}{1+n_{E_L|L}(\beta_2)+n_{E_L|L}(\beta_1)-\mathrm{tr}_{E_L|L}(\beta_1{}^{\sigma}\beta_2)}.$$

Proof. By Lemma 3.4, ${}^{\tau}aa^{-1}\in H_L$. This implies $\rho_L(ax^{\tau}a^{-1}x^{-1})=\rho_L(axa^{-1}x^{-1})$, since $\rho_L({}^{\tau}aa^{-1})=1$. Thus we can prove the first part in the same way as Lemma 2.6. Now we prove the case $a=(1+\beta_1\xi+\beta_2\xi^2)\xi^2$. Put $x=1+y$ for $y\in P_{E_L}^{m'}$. By Lemma 2.5,

$$\begin{aligned}&axa^{-1}\\&=1+\frac{(\beta_1y\xi+\beta_2{}^{\sigma}y\xi^2+{}^{\sigma^2}y)(1-{}^{\sigma}\beta_1{}^{\sigma^2}\beta_2+(\beta_2{}^{\sigma^2}\beta_2-\beta_1)\xi+(\beta_1{}^{\sigma}\beta_1-\beta_2)\xi^2)}{\det(a)}.\end{aligned}$$

Thus we have

$$\begin{aligned}&\rho_L(aga^{-1}g^{-1})\\&=\psi_{E_L}\left(\frac{({}^{\sigma}\gamma-\gamma)(1-{}^{\sigma^2}\beta_1\beta_2)+({}^{\sigma^2}\gamma-\gamma)(n_{E_L|L}(\beta_2)-{}^{\sigma}\beta_1{}^{\sigma^2}\beta_2)}{1+n_{E_L|L}(\beta_2)+n_{E_L|L}(\beta_1)-\mathrm{tr}_{E_L|L}(\beta_1{}^{\sigma}\beta_2)}(x-1)\right).\end{aligned}$$

Therefore it suffices to say that $T(\beta_1,\beta_2)$ is τ-invariant. It follows from ${}^{\tau}\beta_1={}^{\tau}\beta_2\beta_1$, ${}^{\tau}\beta_2\beta_2=1$ and $\gamma\in E$ that

$$\begin{aligned}{}^{\tau}T(\beta_1,\beta_2)&=\frac{({}^{\sigma^2}\gamma-\gamma)(1-{}^{\sigma}\beta_1{}^{\sigma^2}\beta_2/n_{E_L|L}(\beta_2))+({}^{\sigma}\gamma-\gamma)(1-{}^{\sigma^2}\beta_1\beta_2)/n_{E_L|E}(\beta_2)}{(n_{E_L|L}(\beta_2)+n_{E_L|L}(\beta_1)+1-\mathrm{tr}_{E_L|L}(\beta_1{}^{\sigma}\beta_2))/n_{E_L|E}(\beta_2)}\\&=T(\beta_1,\beta_2).\end{aligned}$$

□

Lemma 3.6. *Put* $E^0=\mathrm{Ker}\,\mathrm{tr}_E$ *and* $C'(\mu)=q^{m'-\mu-1+[(m'-\mu-n)/3]+[(n-2\mu-1)/3]}$.

1. *Let* μ *be a positive integer. If* $n\not\equiv 2\mu \mod 3$ *(resp.* $n\equiv 2\mu \mod 3$*), then the map* $x\mapsto S(x)$ *induces a surjection from* $\varpi_E^{\mu}\mathcal{O}_{E_L}^{\times}/1+P_{E_L}^{m'-\mu}$ *to* $(P_E^{2\mu-n}-P_E^{2\mu-n+1})\cap E^0/P_E^{m'+\mu-n}\cap E^0$ *(resp.* $P_E^{2\mu-n+1}\cap E^0/P_E^{m'+\mu-n}\cap E^0$*) and each fiber of the map has* $(q+1)C'(\mu)$ *(resp.* $(q^2-1)C'(\mu)$*) elements.*

2. *For any* $x_0\in\mathcal{O}_{E_L}^{\times}$ *such that* $1+\mathrm{tr}_{L|F}(n_{E_L|L}(x_0))-\mathrm{tr}_{{}^{\sigma^2}E|F}(n_{E_L|{}^{\sigma^2}E}(x_0))\notin P_F$, *the map* $x\mapsto S(x)$ *induces a surjection from* $\{x\in\mathcal{O}_{E_L}^{\times}/1+P_{E_L}^{m'}\,|\,x\equiv x_0 \mod P_{E_L}\}$ *to*

$$\left\{x\in P_E^{-n}\cap E^0/P_E^{m'-n}\cap E^0\;\middle|\;x\equiv S(x_0)\mod P_E^{1-n}\right\},$$

and each fiber of the map has $C'(0)$ *elements.*

3. Fix $\beta_2 \in \mathcal{O}_{E_L}$ such that $n_{E_L|E}(\beta_2) = 1$. Then the map $x \mapsto T(x, \beta_2)$ induces a surjection from $\{x \in P_{E_L} | {}^\tau x = {}^\tau\beta_2 x\}$ to

$$\Big\{x \in P_E^{-n} \cap E^0 / P_E^{m'-n} \cap E^0 \;\Big|\; x \equiv \frac{{}^\sigma\gamma - \gamma + ({}^{\sigma^2}\gamma - \gamma) n_{E_L|L}(\beta_2)}{1 + n_{E_L|L}(\beta_2)} \mod P_E^{1-n}\Big\},$$

and each fiber of the map has $q^{[(m'-n)/3]+[(n-1)/3]}$ elements.

Proof. Since

$$\begin{aligned} S(\beta) &= \frac{\operatorname{tr}_{E_L|E}((\gamma - {}^\sigma\gamma)(n_{E_L|\sigma^2 E}(\beta) - n_{E_L|L}({}^\tau\beta)))}{1 + \operatorname{tr}_{L|F}(n_{E_L|L}(\beta)) - \operatorname{tr}_{\sigma^2 E|F}(n_{E_L|\sigma^2 E}(\beta))} \\ &= \frac{(\gamma - {}^\sigma\gamma)(n_{E_L|\sigma^2 E}(\beta) - n_{E_L|L}({}^\tau\beta)) - {}^{\sigma^2}((\gamma - {}^\sigma\gamma)(n_{E_L|\sigma^2 E}(\beta) - n_{E_L|L}({}^\tau\beta)))}{1 + \operatorname{tr}_{L|F}(n_{E_L|L}(\beta)) - \operatorname{tr}_{\sigma^2 E|F}(n_{E_L|\sigma^2 E}(\beta))}, \end{aligned}$$

$S(\beta)$ belongs to E^0. When $v_{E_L}(\beta) = \mu$, we have $S(\beta) \in P_E^{2\mu-n}$, $P_E^{2\mu-n} \cap E^0 = P_E^{2\mu-n+1} \cap E^0$ if $n \not\equiv 2\mu \mod 3$ and

$$\begin{aligned} &S(\beta(1+x)) \\ &\quad\equiv S(\beta) + \operatorname{tr}_{E_L|E}((\gamma - {}^\sigma\gamma) n_{E_L|\sigma^2 E}(\beta) \operatorname{tr}_{E_L|\sigma^2 E}(x)) \mod P_E^{\mu-n+v_{E_L}(x)+1}. \end{aligned}$$

for $x \in P_{E_L}$. Therefore $x \mapsto S(x)$ induces a well-defined map from

$$\varpi_E^\mu \mathcal{O}_{E_L}^\times / (1 + P_{E_L}^{m'-\mu})$$

to

$$\begin{cases} (P_E^{2\mu-n} - P_E^{2\mu-n+1}) \cap E^0 / P_E^{m'+\mu-n} \cap E^0 & \text{if} \quad n \not\equiv 2\mu \mod 3, \\ P_E^{2\mu-n+1} \cap E^0 / P_E^{m'+\mu-n} \cap E^0 & \text{if} \quad n \equiv 2\mu \mod 3. \end{cases}$$

Denote the induced map by $S_{m'-\mu}$ and put

$$c_i = \begin{cases} (q+1) q^{i-1+[(2m'-i-n)/3]+[(n-2m'+2i-1)/3]} & n \not\equiv 2\mu \mod 3, \\ (q^2-1) q^{i-1+[(2m'-i-n)/3]+[(n-2m'+2i-1)/3]} & n \equiv 2\mu \mod 3. \end{cases}$$

First assume $\mu > 0$. We prove the assertion by induction on $m' - \mu$. Let $m' - \mu = 1$. When $n \equiv 2\mu \mod 3$, $P_E^{2\mu-n+1} \cap E^0 / P_E^{m'+\mu-n} = P_E^{2m'-n-1} \cap E^0 / P_E^{2m'-n-1} \cap E^0$. Since $|\varpi_E^\mu \mathcal{O}_{E_L}^\times / (1 + P_{E_L})| = q^2 - 1$, S_1 is surjective and its fiber has $q^2 - 1$ elements. Now assume $n \not\equiv 2\mu \mod 3$. Since

$$S(x) \equiv (\gamma - {}^\sigma\gamma) n_{E_L|\sigma^2 E}(x) - {}^{\sigma^2}((\gamma - {}^\sigma\gamma) n_{E_L|\sigma^2 E}(x)) \mod P_E^{2\mu-n+1} \cap E^0$$

and $n_{E_L|\sigma^2 E}$ induces a surjective homomorphism from $\mathcal{O}_{E_L}^\times / (1 + P_{E_L})$ to $\mathcal{O}_{\sigma^2 E}^\times / (1 + P_{\sigma^2 E})$, it follows that S_1 is a surjection and each fiber of S_1 has $q+1$ elements. Let $m' - \mu > 1$ and take any element $y \in (P_E^{2\mu-n} - P_E^{2\mu-n+1}) \cap E^0$. By the induction assumption, there exist $c_{m'-\mu-1}$ elements $x \in \varpi_E^\mu \mathcal{O}_{E_L}^\times / 1 + P_{E_L}^{m'-\mu-1}$ such that $S_{m'-\mu-1}(x) = y \mod P_E^{m'+\mu-n-1}$. Put $x' = x(1 + x_1)$ for $x_1 \in P_{E_L}^{m'-\mu-1}$. Then

$$\begin{aligned} S(x') &\equiv S(x) + \operatorname{tr}_{E_L|E}((\gamma - {}^\sigma\gamma) n_{E_L|\sigma^2 E}(x)(n_{E_L|\sigma^2 E}(1 + x_1) - 1)) \\ &\qquad\qquad \mod P_E^{m'+\mu-n} \cap E^0 \\ &\equiv S(x) + (\gamma - {}^\sigma\gamma) n_{E_L|\sigma^2 E}(x) \operatorname{tr}_{E_L|\sigma^2 E} x_1 \\ &\qquad - {}^{\sigma^2}((\gamma - {}^\sigma\gamma) n_{E_L|\sigma^2 E}(x) \operatorname{tr}_{E_L|\sigma^2 E} x_1) \mod P_E^{m'+\mu-n} \cap E^0. \end{aligned}$$

Since $x_1 \mapsto (\gamma - {}^{\sigma}\gamma)n_{E_L|^{\sigma^2}E}(x)\operatorname{tr}_{E_L|^{\sigma^2}E} x_1 - {}^{\sigma^2}((\gamma - {}^{\sigma}\gamma)n_{E_L|^{\sigma^2}E}(x)\operatorname{tr}_{E_L|^{\sigma^2}E} x_1)$ induces a surjective k_F linear map from $P_{E_L}^{m'-\mu-1}/P_{E_L}^{m'-\mu}$ to $P_E^{m'+\mu-n-1} \cap E^0/P_E^{m'+\mu-n} \cap E^0$, it follows that $S_{m'-\mu}$ is surjective and each fiber of it has

$$c_{m'-\mu} = \begin{cases} qc_{m'-\mu-1} & \text{if} \quad m'+\mu-n-1 \not\equiv 0 \mod 3, \\ q^2 c_{m'-\mu-1} & \text{if} \quad m'+\mu-n-1 \equiv 0 \mod 3 \end{cases}$$

elements. The case $\mu = 0$ can be proved in the same way. For the map $x \mapsto T(x, \beta_2)$, we can prove our assertion in the same way as the proof of Lemma 2.7 since

$$T(x,\beta_2) = \frac{{}^{\sigma}\gamma - \gamma + ({}^{\sigma^2}\gamma - \gamma)n_{E_L|L}(\beta_2)}{1 + n_{E_L|L}(\beta_2)} + \frac{(\gamma - {}^{\sigma}\gamma)^{\sigma^2} x\beta_2 - {}^{\sigma^2}((\gamma - {}^{\sigma}\gamma)^{\sigma^2} x\beta_2)}{1 + n_{E_L|L}(\beta_2)}.$$

□

By the above three lemmas, we have the following lemma. See also the proof of Lemma 2.8 and Proposition 2.9.

Lemma 3.7. 1. *Let $0 < \mu < m$ and $n \not\equiv 2\mu \bmod 3$. For $x \in \mathcal{O}_{E_L}^{\times}$,*

$$\sum_{\substack{a = 1 + \beta\xi + {}^{\tau}\beta\xi^2 \\ \beta \in \varpi_E^{\mu}\mathcal{O}_{E_L}^{\times}/1 + P_{E_L}^{m'-\mu}}} \rho_L(ax^{\tau}a^{-1}x^{-1})$$
$$= \begin{cases} 0 & \textit{if} \quad n_{E_L|E}(x) \notin U_{n-2\mu}, \\ -(q+1)q^{2(m'-\mu-1)} & \textit{if} \quad n_{E_L|E}(x) \in U_{n-2\mu}^{*}, \\ (q^2-1)q^{2(m'-\mu-1)} & \textit{if} \quad n_{E_L|E}(x) \in U_{n+1-2\mu}. \end{cases}$$

2. *Let $0 < \mu < m$ and $n \equiv 2\mu$ mod 3. For $x \in \mathcal{O}_{E_L}^{\times}$,*

$$\sum_{\substack{a = 1 + \beta\xi + {}^{\tau}\beta\xi^2 \\ \beta \in \varpi_E^{\mu}\mathcal{O}_{E_L}^{\times}/1 + P_{E_L}^{m'-\mu}}} \rho_L(ax^{\tau}a^{-1}x^{-1})$$
$$= \begin{cases} 0 & \textit{if} \quad n_{E_L|E}(x) \notin U_{n-2\mu}, \\ (q^2-1)q^{2(m'-\mu-1)} & \textit{if} \quad n_{E_L|E}(x) \in U_{n+1-2\mu}. \end{cases}$$

3. *For $x \in \mathcal{O}_{E_L}^{\times}$,*

$$\sum_{\substack{a = 1 + \beta\xi + {}^{\tau}\beta\xi^2 \\ \beta \in \mathcal{O}_{E_L}^{\times}/(1 + P_{E_L}^{m'})}} \rho_L(ax^{\tau}a^{-1}x^{-1}) = 0 \quad \textit{if} \quad n_{E_L|E}(x) \notin U_n.$$

4. *Fix $\beta_2 \in \operatorname{Ker} n_{E_L|E}$. For $x \in \mathcal{O}_{E_L}^{\times}$,*

$$\sum_{\substack{a = (1 + \beta_1\xi + {}^{\tau}\beta\xi^2)\xi^2 \\ \beta_1 \in \{y \in P_{E_L}/P_{E_L}^{m'} \ | \ {}^{\tau}y = {}^{\tau}\beta_2 y\}}} \rho_L(ax^{\tau}a^{-1}x^{-1}) = 0 \quad \textit{if} \quad n_{E_L|E}(x) \notin U_n.$$

By this lemma, we can calculate χ_{κ_θ} on $\mathcal{O}_E^{\times} - U_n$. On U_n^{*}, it is already calculated in Lemma 2.10. By Theorem 1.8, it gives χ_{π_θ} on $\mathcal{O}_E^{\times} - U_{n+1}$. On U_{n+1}, it is given in the proof of Theorem 2.13. It remains to calculate χ_{κ_θ} on $E^{\times} - U_1$.

Lemma 3.8. *For* $x \in E^\times - U_1$,

$$\chi_{\kappa_\theta}(x) = (-1)^{n+1}\theta(x).$$

Proof. By (1.3) and (1.5), it suffices to show that

$$\chi_{\mathrm{Ind}_H^B \rho_\theta}(x) = \theta(x).$$

Since

$$\chi_{\mathrm{Ind}_H^B \rho_\theta}(x) = \sum_{a \in H\backslash B} \rho_\theta(axa^{-1}),$$

we have only to show that if $axa^{-1} \in H$ for $a \in B$, then $a \in H$.

Let R be a natural ring morphism from A^0 to k_F^3 by the identification A^0/A^1 with k_F^3. We note that if $R(a) = (\alpha_0, \alpha_1, \alpha_2)$, then $R(\varpi_E a \varpi_E^{-1}) = (\alpha_1, \alpha_2, \alpha_0)$ and if $a \in E$, then $\alpha_0 = \alpha_1 = \alpha_2$. We may assume $a \in A^0$. Let $R(a) = (\alpha_0, \alpha_1, \alpha_2)$ and $x = \varpi_E^i x_0$ for $x_0 \in \mathcal{O}_E^\times$. Since $x \notin F^\times(1 + P_E)$,

$$R(axa^{-1}x^{-1}) = \begin{cases} (\alpha_0\alpha_1^{-1}, \alpha_1\alpha_2^{-1}, \alpha_2\alpha_0^{-1}) & \text{if} \quad i \equiv 1 \mod 3, \\ (\alpha_0\alpha_2^{-1}, \alpha_1\alpha_0^{-1}, \alpha_2\alpha_1^{-1}) & \text{if} \quad i \equiv 2 \mod 3. \end{cases}$$

Therefore $axa^{-1}x^{-1} \in E^\times K^1$ implies $\alpha_0\alpha_1^{-1} = \alpha_1\alpha_2^{-1} = \alpha_2\alpha_0^{-1}$ or $\alpha_0\alpha_2^{-1} = \alpha_1\alpha_0^{-1} = \alpha_2\alpha_1^{-1}$. In any case, $\alpha_0 = \alpha_1 = \alpha_2$, since $k_E = k_F$ has no cube root of unity. Thus $a \in E^\times K^1$. Now we may assume $a - 1 \in A^j - (P_E^j + A_E^{j+1})$ for $j \geq 1$. Put $a - 1 = \varpi_E^j a_0$ and $R(a_0) = (\beta_0, \beta_1, \beta_2)$. Since

$$\begin{aligned} axa^{-1}x^{-1} &= 1 + (a-1) - x(a-1)x^{-1} \mod A^{j+1} \\ &= 1 + \varpi_E^j(a_0 - xa_0x^{-1}) \mod A^{j+1}, \end{aligned}$$

$R(a_0 - xa_0x^{-1}) = (\beta_0 - \beta_1, \beta_1 - \beta_2, \beta_2 - \beta_0)$ or $(\beta_0 - \beta_2, \beta_1 - \beta_0, \beta_2 - \beta_1)$. Therefore $axa^{-1}x^{-1} \in E^\times K^{j+1}$ contradicts $a - 1 \in A^j - (P_E^j + A^{j+1})$. This implies $a \in E^\times K^m$ if $axa^{-1}x^{-1} \in E^\times K^m$. □

Now we can state the character formula of π_θ for the non-Galois case. We remark that there is no ramified extension of F with degree 3 which is not isomorphic to E.

Theorem 3.9. *Let* $q \equiv 2 \bmod 3$, *and let the other notation be as in Theorem 2.13. Let* x *be a regular elliptic element in* $\mathrm{GL}_3(F)$.

1. If $F(x)|F$ *is unramified, then*

x	$\chi_{\pi_\theta}(x)$
$x \notin F^\times(1 + P_{F(x)}^{n+1})$	0
$c(1+y)(c \in F^\times, y \in P_{F(x)}^{n+1})$	$q^{n-1}(q^2+q+1)\theta(c)$

2. If $x \in E$, *then*

x	$\chi_{\pi_\theta}(x)$
$x \in E^\times - U_0$	$(-1)^{n+1}\theta(x)$
$x \in U_j^* \quad (1 \leq j \leq n-1)$	$(-1)^{n+1}(-q)^j\theta(x)$
$c(1+y) \in U_n^*\ (c \in F^\times, y \in \varpi_E^n \mathcal{O}_E^\times)$	$q^{n-1}\theta(c)\,\mathrm{Kl}(\gamma y)$
$x \in U_{n+1}$	$q^{n-1}(q^2+q+1)\theta(x)$

where the Kloosterman sum $\mathrm{Kl}(a)$ *is defined by* (2.17).

Proof. The character formula for the unramified case is obtained in the same way as the Galois case. By Proposition 3.3, (3.4), Lemma 3.4, Lemma 3.7 and Lemma 3.8, we get for $x \in E^\times - U_{n+1}$,

$$\chi_{\kappa_\theta}(x) = \begin{cases} (-1)^{n+1}\theta(x) & \text{if} \quad x \in E^\times - U_0, \\ (-1)^{n+1}(-q)^j\theta(x) & \text{if} \quad x \in U_j^* \quad (1 \leq j \leq n-1). \end{cases}$$

On U_n^*, this is obtained by Lemma 2.10. By Theorem 1.8,

$$\chi_{\pi_\theta}(x) = \chi_{\kappa_\theta}(x) \quad \text{for } x \in E^\times - U_{n+1}.$$

On U_{n+1}, the proof of Theorem 2.13 holds for the non-Galois case. Thus we get the character formula for π_θ on $E^\times$. □

Acknowledgement

The author would like to express his sincere gratitude to the referee for much helpful advice.

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